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Sets, Trees and Incompleteness

door

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Preface

This master's thesis is the written result of work that I executed in the academic year 2017-2018. I would like to express my deep gratitude to Andreas Weiermann for being my guide through unknown waters. I would also like to sincerely thank Michael Drmota (Technische Universität Wien) and Ali Enayat (University of Gothenburg) for their helpful comments. Special thanks to my parents and my sister for their encouragements.

Ben De Bondt, May 2018

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Introduction

As a consequence of Kurt Gödel's first incompleteness theorem, proved in 1931, theories such as PA and ZFC are, if consistent, incomplete. This means that there are sentences in the language of first-order arithmetic (respectively the language of first-order set theory) that are independent from the PA-axioms (respectively ZFC-axioms)[1]. As regards the theory PA, for more than forty years, it was not clear whether any such independent sentences could possibly come up in research in any of the classical areas not directly related to mathematical logic such as combinatorics, number theory, algebra, analysis, geometry, ... All this while, the possibility remained that the only way of producing PA-independent sentences was to code logical concepts, such as provability or consistency into a number-theoretic formula, thus creating formulas that are somehow unnatural to the areas described above.

This turned out not to be the case when Jeff Paris and Leo Harrington proved in 1972 that the strengthened finite Ramsey principle, a natural statement in finite Ramsey theory, is unprovable in PA. The seventies and eighties subsequently witnessed the appearance of many other natural arithmetical statements independent of PA: for example the termination of Goodstein sequences (Kirby and Paris (1982)), Kirby's flipping principle (Kirby (1982)), the Kanamori-McAloon Theorem (Kanamori and McAloon (1987)) and several examples in well-quasi-order theory (for instance Simpson (1985)). We refer to the overview in Bovykin (2009) and to the course *Capita Selecta in de Logica* for many more examples.

The original proofs of these independence results were mostly model theoretic in nature[2], and in particular centered around the analysis of initial segments of models of PA and their indicators. It was not long before also proof theoretic methods, in particular the classification of the provably total recursive functions in various theories[3], proved very useful in the study of independence results, today even surpassing the method of indicators in popularity[4].

Upon today, both model theoretic and proof theoretic approaches are used to prove independence results and both have their own strengths and weaknesses. This proof-model duality is general and originates in the completeness theorem of first order logic that entails that one can think about unprovability of φ in

[1] As customary, we tend to omit the clause "provided that ZFC/PA is consistent".

[2] Model theory had become to blossom as an independent field in the period between 1931 and 1972. There is no coincidence in the fact that this period falls entirely in the period of active work of Alfred Tarski.

[3] See the course *Bewijstheorie*.

[4] Cf. Kaye (1991) page 193.

two equivalent ways: as non-existence of a proof of φ or as existence of a model for $\neg\varphi$.

Using the method of ordinal analysis, Andreas Weiermann established a renewed connection between incompleteness and various parts of mathematics seemingly non-related to mathematical logic, by means of his program of phase transitions. A particularly striking theorem in this realm is the following.

Theorem 0.1 (Weiermann (2003))
Let for any function $f : \omega^2 \to \omega$, which is definable in the language of first-order arithmetic, $\varphi(f)$ be the corresponding miniaturisation sentence of Kruskal's theorem. For any rational $r \in \mathbb{Q}$, let $f_r : \omega \to \mathbb{Q}$ be defined by $f_r(k, l) = k + r|l|$, where $|l|$ denotes the length of the binary representation of l. Then provability of the statement $\varphi(f_r)$ depends on $r \in \mathbb{Q}$ in the following way.

- If $r < \dfrac{\log 2}{\log \sigma}$, then $\varphi(f_r)$ is provable in PA.
- If $r > \dfrac{\log 2}{\log \sigma}$, then $\varphi(f_r)$ is independent from PA.

Here, σ is Otter's constant (see Chapter 3).

The proof of this theorem draws on results from analytical combinatorics, in particular the work of Otter (Otter (1948)).

In this thesis we will explore to what extent certain techniques developed for proving incompleteness results for PA and related theories can be applied to prove analogous results for set theories. We focus first on finding an analogue for Weiermann's Theorem 0.1 for a weak set theory Sim. This will take up the first five chapters of the thesis. It entails extending the work of Otter to determine the asymptotics of the counting functions of certain families of trees, which we accomplish in Chapter 3. Combining this asymptotic information with proof theoretic techniques, we can indeed derive in Chapter 5 a transition from provability to independence just like the one in Theorem 0.1.
What makes this approach work, is that the set theory Sim is proof theoretically weak. This, in the sense that its proof theoretical ordinal, Γ_0, is of moderate size and therefore easily understood combinatorically.

In the last three chapters, we concentrate on stronger set theories, such as ZFC and above. These are currently way out of reach of the proof theoretical methods we employ for the theory Sim. It is a notorious open problem in proof theory to determine the proof theoretic ordinal of ZFC. Already the proof theoretical analysis of much weaker subsystems of second order arithmetic poses overwhelming technical complications[5] and this is illustrative for the status of the problem of the proof theoretical ordinal of ZFC as currently unreachable. It is therefore all the more interesting that the Kirby-Paris-method of indicators can be applied to stronger set theories on the level of ZFC. This possibility was brought forward in McAloon and Ressayre (1981).

[5] See for instance Rathjen (2000).

The method of initial segments and their indicators did however receive much less attention in the context of set theory than arithmetic. The reason for this is clear: with the dawn of the technique of forcing in the 1960's, set theory had already been supplied with a multitude of set theoretically natural independent statements. There has however always been an active tradeoff between the studies of the theories PA and ZFC[6].

In Chapters 6 and 7 we will take up this model theoretic theme and study end extensions of models of arithmetic and set theory. We conclude the thesis in Chapter 8 by giving a quicker overview of some selected results in McAloon and Ressayre (1981), in particular their construction of an indicator for so called "strong cuts" in models of ZFC-set theory.

We list the main innovative parts of this thesis.

- In Chapter 3, we determine the asymptotical behavior of families of trees that seem not yet to have come up in literature (the sequences are also currently unknown to the OEIS).
- We deduce a new incompleteness result for the set theory Sim, related to the result in Weiermann (2003).

Moreover, the proof of Theorem 7.7 in Chapter 7 is our own detailed elaboration of a proof sketch in McAloon and Ressayre (1981). A vital step in this proof was supplied to us by Ali Enayat. We did hitherto not find a complete proof of this theorem in literature.

[6]For example, in the classic paper Kirby and Paris (1977), the study of semi-regular, regular and strong initial segments is motivated by drawing parallels with large cardinal theory. Also, the technique of forcing is now frequently used to construct models of arithmetic.

Errata

- The compactness argument mentioned on page 43 wich proves that $S(z)$ has an analytic extension to a Δ-region $\Delta(R, \phi)$ should be deferred to the proof of the next Lemma 3.6 which should read instead

  Lemma 3.6
  Let $S(z)$ be analytic on $B(0, \rho)$, having non-negative MacLaurin coefficients and satisfying $F(z, S(z)) = 0$ on $\overline{B}(0, \rho)$, where $F(z, y)$ is a function analytic in both variables at the point $(\rho, S(\rho))$.
  Suppose that

  - $F(\rho, S(\rho)) = \partial_y F(\rho, S(\rho)) = 0$,
  - $\partial_{yy} F(\rho, S(\rho)) > 0$ and
  - $\partial_z F(\rho, S(\rho)) > 0$.

  Then $S(z)$ allows an analytic extension to a Δ-region $\Delta(R, \phi)$ with inlet at ρ. In addition, there is $c \in \mathbb{R}_{>0}$, $d \in \mathbb{R}$ such that

  $$S(z) = S(\rho) - c\sqrt{\rho - z} + d(\rho - z) + O((\rho - z)^{3/2}),$$

  holds on $\Delta(R, \phi)$.

  In fact, c is given explicitly by $c = \sqrt{\dfrac{2\,\partial_z F(\rho, S(\rho))}{\partial_{yy} F(\rho, S(\rho))}}$.

  After we have determined in the proof of this lemma the local square root behaviour of $S(z)$, the described compactness argument can indeed be carried out to show that $S(z)$ has an analytic extension to a Δ-region $\Delta(R, \phi)$, thus finishing the proof of this lemma.
  On page 48, the sentence "Hence, $A(z)$ can be analytically continued to a Δ-region with inlet at ρ." can then be deleted.

- On page 46, the third assertion of Theorem 3.6 should read $\rho_{n-1} > \rho_n$, in the proof it should correspondingly read $\rho_n \geq \rho_{n+1}$.

- We would like to stress our gratitude towards Michael Drmota for helping us overcome some difficulties that we experienced in proving Theorem 3.6. In particular we would like to clarify that the proof of Theorem 3.6 as it is presented on page 46 is based on an argument that he generously shared with us in personal communication.

- On page 74, the last line should read "Consider the tree of strictly decreasing $(g + n)$-bounded finite sequences of Γ-trees".

- On page 75, on the third last line, it should read $f_r : \omega \to \mathbb{R}$.

Contents

1

Systems of set theory

1.1 Introduction

We intend to study set theories and their models.
First of all, let us formulate some general remarks.

We will take the viewpoint that the following question:

Question 1

What are sets and what are their properties?

is not the right one to ask. Indeed, we favour the question:

Question 2

What are models of set theory and what are their properties?

This may seem strange, but is in fact a situation often encountered in modern mathematics. A familiar example is to be found in a common branch of mathematics such as Linear Algebra, where one does not try to define nor study specific *vectors*, but indeed studies a specific axiomatisation of the concept of *vector space*.

However, even Question 2 is meaningless as long as we do not fix a specific set theory whose models are to be studied. In contrast to the situation most common in Linear Algebra, there is an abundance of possible interesting set theories.
Hence, we set it as our goal to study these set theories and their models.

Starting in 1934 with the first construction of a non-standard model of arithmetic by Thoralf Skolem[1], a theory emerged which is still in want of a definitive name[2], but may be denoted as "the study of models of Peano Arithmetic". Clearly, the ethos of this theory is in perfect agreement with the philosophy just outlined: instead of *numbers*, the central objects of study are *models of number theory*. We will use the study of models of Peano Arithmetic to guide us and help us select specific topics in the study of models of set theories, which is clearly too broad to be contained in one thesis. Indeed, in this thesis we venture to discuss how some selected results and techniques for Peano Arithmetic can be adapted for set theories.

1.2 Set Theories

Extensionality

First, we fix the language of set theory $\mathcal{L}_\in$. Let $\mathcal{L}_\in$ be the first-order logical language containing the 2-ary relation symbol "$\in$" as its sole non-logical symbol. In this language, one can formulate axioms such as:

Axiom of Extensionality

$$\forall x, y \quad \forall z(z \in x \Leftrightarrow z \in y) \Rightarrow (x = y).$$

For our purposes, a set theory T will be a first-order theory in the language $\mathcal{L}_\in$ (or an expansion thereof) that contains the axiom of extensionality. This definition is extremely general as it includes as models $\mathcal{M} = (M, \in^\mathcal{M})$ of possible set theories all sets endowed with an existential relation. As we can see in the following example, this includes structures one might not consider very well-suited as models of set theory.

Example 1.1
The $\mathcal{L}_\in$-structures

$$\mathcal{M} = (\mathbb{R}, \in^\mathcal{M} = \{(r, s) \in \mathbb{R}^2 : r < s\})$$

and

$$\mathcal{M}_+ = (\mathbb{R}_{\geq 0}, \in^{\mathcal{M}_+} = \{(r, s) \in \mathbb{R}^2_{\geq 0} : r < s\})$$

are $\mathcal{L}_\in$-structures satisfying the axiom of extensionality.

In these example, the *sets* are real numbers. One of the curiosities of the first model is that it contains no empty set:

$$\mathcal{M} \models (\forall x)(x \neq \emptyset).$$

[1]See Skolem (1934)

[2]Kirby remarks in Kirby (1992): "Our vocabulary lacks a term to denote a person whose calling is the study of models of arithmetic. Model theorists, topologists, even functional analysts can identify themselves succinctly, but we have to resort to such locutions as "I'm in models of arithmetic." And the name of the field itself – "models of arithmetic" – also seems to bespeak an insecurity about whether it is a field at all: the objects of study are baldly named without any pretensions to a grand theory or -ology. Models of arithmetic certainly is a bona fide field. It has its own meetings, folklore, and stars. It has built up a coherent body of knowledge relevant to some of the central problems of modern logic."

The model $\mathcal{M}_+$ contains an empty set:

$$\mathcal{M}_+ \models 0 = \emptyset,$$

but has no set containing only the empty set:

$$\mathcal{M}_+ \models (\forall x)(x \neq \{\emptyset\}).$$

An illustration that we should not expect these very weakest of set theories to capture our intuition on the nature of sets.

Well-known set theories include the system KP of Kripke-Platek set theory, the systems ZF and ZFC of Zermelo-Fraenkel set theory, with or without choice, and the system, GB of Godel-Bernays set theory. In this chapter we will also introduce weaker set theories, that in particular do not contain the powerset axiom, nor replacement/collection schemes, but are in turn strong enough to capture parts of mathematical intuition.

We gradually introduce the axioms and notions involved. This is necessary to fix our notations and conventions. Along the way we will assume the reader is already familiar with the axioms of ZF set theory and their use, in order to speed up our introduction.

Infinity

One of the main objectives of set theory is to study the properties of the notion of "Infinity". For this reason, many set theories contain an axiom postulating the existence of a specific infinite set. We formalize the axiom of infinity in the usual way, i.e. via the existence of an inductive set. A set x is called inductive if it contains the empty set and is closed under the operation $y \mapsto y \cup \{y\}$. We formalize this in $\mathcal{L}_\in$ as follows:

$$\begin{aligned}\mathrm{Ind}(x) \equiv & [(\exists y)((y \in x) \wedge (\forall z)(z \notin y))] \\ & \wedge [(\forall y)(y \in x \Rightarrow (\exists z)((z \in x) \wedge (\forall a)(a \in z \iff (a \in y \vee a = y))))] \,.\end{aligned}$$

The axiom then reads:

Axiom of Infinity

$$(\exists x)\, \mathrm{Ind}(x).$$

Class terms

In order to avoid spelling out even longer $\mathcal{L}_\in$-formulas, we extend the language $\mathcal{L}_\in$ to a language $\mathcal{L}_\in^{\mathrm{Class}}$, by adding class terms, together with the usual rules governing their use. We allow for nested class terms and terms can contain free variables. Theorems on eliminability of class terms and conservation of provability between $\mathcal{L}_\in$ and $\mathcal{L}_\in^{\mathrm{Class}}$ can be obtained in the usual way. See for example Levy (1979). We will also employ the standard notation for specific classes such as ω, On, V, ... and so on.

Levy's hierarchy

$\mathcal{L}_\in$-formula's range from very simple, e.g.

$$\varphi_1(v_1, v_2) \equiv v_1 = v_2,$$

to more complex, e.g.

$$\varphi_2(v_1, v_2) \equiv \exists y \forall x (\forall z \in y)((x \in y \Leftrightarrow x \in v_1 \vee x \in v_2) \wedge (z \in v_1)).$$

To quantify the complexity of $\mathcal{L}_\in$-formulas induced by the quantifiers, the Σ_i, Π_i and Δ_i hierarchies are introduced.

Definition 1.1
The set of Δ_0-formulas or bounded formulas (of $\mathcal{L}_\in$) is inductively defined by

1. Every atomic formula ($x \in y$, $x = y$) is a Δ_0-formula.
2. If φ_1, φ_2 are Δ_0-formulas, then also $\neg\varphi_1$, $\varphi_1 \wedge \varphi_2$, $\varphi_1 \vee \varphi_2$ are Δ_0-formulas.
3. If φ is a Δ_0-formula and x, y are two different variables, then $(\forall x \in y)(\varphi)$ and $(\exists x \in y)(\varphi)$ are Δ_0-formulas.

An $\mathcal{L}_\in$-formula is Σ_0 if and only if it is Π_0 if and only if it is Δ_0.
An $\mathcal{L}_\in$-formula φ is Σ_{n+1} (respectively Π_{n+1}) if and only if there is a Π_n-formula (respectively Σ_n-formula) ψ such that $\varphi \equiv \exists v \psi$ (respectively $\varphi \equiv \forall v \psi$).
For an $\mathcal{L}_\in$-theory T, an $\mathcal{L}_\in$-formula φ is Σ_n^T (respectively Π_n^T) if and only if there is an Σ_n-formula (respectively a Π_n-formula) ψ such that $T \vdash \varphi \iff \psi$.

Example 1.2
One may verify that the formula $\varphi_2(v_1, v_2)$ defined above is both Σ_2 and $\Delta_0^{\mathsf{Extensionality}}$.
(Hint: check that $\mathsf{Extensionality} \vdash (\forall v_1, v_2)(\varphi_1(v_1, v_2) \Leftrightarrow \varphi_2(v_1, v_2))$.)

Definition 1.2
For an $\mathcal{L}_\in$-theory T a class $A(v_1, \ldots, v_k)$ is a Δ_0^T-class if there exists a Δ_0^T-$\mathcal{L}_\in$-formula $\varphi(v_0, v_1, \ldots, v_k)$ such that

$$T \vdash (\forall x_1, \ldots, x_k)(A(x_1, \ldots, x_k) = \{x : \varphi(x, x_1, \ldots, x_k)\}).$$

Δ_0-separation

Δ_0-Separation Scheme

This scheme asserts for every Δ_0-formula φ, with $y \notin FV(\varphi)$, that

$$(\forall x)(\exists y)(y = \{z : (z \in x) \wedge \varphi\}).$$

Already this weakest of separation schemes is strong enough to show that V is not a set (the formula $x \notin x$ is Δ_0).

Rudimentary closure

In absence of the strong set construction axioms from ZF set theory, one needs to explicitly stipulate for certain classes that they are sets. Our rudimentary sets will be generated starting from the following finite list of terms.

- $R_1(v_1, v_2) \equiv \{v_1, v_2\}$
- $R_2(v_1, v_2) \equiv v_1 \setminus v_2$
- $R_3(v_1, v_2) \equiv v_1 \times v_2$
- $R_4(v_1) \equiv \bigcup v_1$
- $R_5(v_1, v_2) \equiv \{(x, y) : (x \in y) \wedge (x \in v_1) \wedge (y \in v_2)\}$
- $R_6(v_1) \equiv \mathrm{dom}(v_1)$
- $R_7(v_1) \equiv {v_1}^{-1}$
- $R_8(v_1) \equiv \{((x, z), y) : ((x, y), z) \in v_1\}$
- $R_9(v_1) \equiv \{((z, x), y) : ((x, y), z) \in v_1\}$
- $R_{10}(v_1, v_2) \equiv v_2[v_1]$

Axiom of Rudimentary Closure

$$(\forall u)(\forall v)\,(\{\,R_1(u, v), R_2(u, v), R_3(u, v), R_4(u), R_5(u), R_6(u), R_7(u), R_8(u), R_9(u), R_{10}(u, v)\} \subseteq V\,)\,.$$

We inductively define the set of rudimentary terms:

- the variables $v_1, v_2, \dots$ are rudimentary terms,
- the terms $R_1(v_1, v_2), \dots, R_{10}(v_1, v_2)$ are rudimentary,
- if $t, t_1, \dots, t_n$ are rudimentary then $t[v_1/t_1, \dots, v_n/t_n]$ is a rudimentary term,
- if t, t_1 are terms, with t_1 rudimentary, and if $\mathsf{Extensionality} \vdash t = t_1$, then t is rudimentary.

Example 1.3
The term $t(x, y) = x \cap y$ is rudimentary. Indeed

$$\mathsf{Extensionality} \vdash x \cap y = x \setminus (x \setminus y).$$

Lemma 1.1
If $t(v_1, \dots, v_n)$ is a rudimentary term, then

$$\mathsf{Extensionality} + \mathsf{Rudimentary\ Closure} \vdash (\forall x_1 \dots, x_n)(t(x_1, \dots, x_n) \in V).$$

Proof
By induction on the term t. □

We prove next that all sets that are forced to exist by the Δ_0-separation axiom can also be constructively created using rudimentary functions.

Theorem 1.1
For every Δ_0-formula $\varphi(u, v_0, v_1, \ldots, v_n)$ (all variables indicated), there is a rudimentary term $t(v_0, v_1, \ldots, v_n)$ such that

$$\mathsf{Extensionality} \vdash (\forall x_0, x_1, \ldots, x_n)(t(x_0, x_1, \ldots, x_n) = \{u : u \in x_0 \wedge \varphi(u, \overline{x})\}).$$

Proof
We show that it suffices to prove the next-following Lemma 1.2. Using this lemma, we find a rudimentary term s such that

$$(\forall y_0, y_1, \ldots, y_{n+1})(s(y_0, y_1, \ldots, y_{n+1}) = \{(u_0, u_1, \ldots, u_{n+1}) : \\ u_0 \in y_0, u_1 \in y_1, \ldots, u_{n+1} \in y_{n+1} \wedge \varphi(u_0, u_1, \ldots, u_{n+1})\}.$$

Now we can choose

$$t(v_0, v_1, \ldots, v_n) = \operatorname{dom}(\operatorname{dom}(\ldots \operatorname{dom}(s(v_0, \{v_0\}, \{v_1\}, \ldots, \{v_n\}))\ldots)).$$

Indeed,

$$\begin{aligned} u \in t(x_0, \ldots, x_n) &\iff (\exists u_1, \ldots, u_{n+1}) u \in x_0, u_1 \in \{x_0\}, \ldots, u_{n+1} \in \{x_n\} \\ &\qquad \wedge \varphi(u, u_1, \ldots, u_n, u_{n+1}) \\ &\iff u \in x_0 \wedge \varphi(u, x_0, \ldots, x_n). \end{aligned}$$

□

Lemma 1.2
For every Δ_0-formula $\varphi(v_1, \ldots, v_n)$ (all variables indicated), there is a rudimentary term $t(v_1, \ldots, v_n)$ such that

$$\mathsf{Extensionality} \vdash (\forall x_1, \ldots, x_n)(t(x_1, \ldots, x_n) = \{(u_1, \ldots, u_n) : \\ u_1 \in x_1, \ldots, u_n \in x_n \wedge \varphi(u_1, \ldots, u_n)\}.$$

Proof
This is a rather technical venture.
We first prove the following claim.

Claim: If $\varphi(x_1, x_2)$ is an $\mathcal{L}_\in$-formula and $t(x_1, x_2)$ a rudimentary term such that

$$(\forall x_1, x_2)(t(x_1, x_2) = \{(u_1, u_2) : u_1 \in x_1, u_2 \in x_2 \wedge \varphi(u_1, u_2)\}).$$

Then, for every $n \geq 2$, for every $1 \leq i \neq j \leq n$ there is a rudimentary term s such that

$$(\forall x_1, \ldots, x_n)(s(x_1, \ldots, x_n) = \{(u_1, \ldots, u_n) : u_1 \in x_1, \ldots, u_n \in x_n \wedge \varphi(u_i, u_j)\}).$$

We prove this by induction on n.
The induction base $n = 2$ is easily checked since in this case we can always choose

either $s(x_1, x_2) = t(x_1, x_2)$ or $s(x_1, x_2) = t(x_2, x_1)^{-1}$ and both are rudimentary. Now suppose $n > 2$.
By changing $\varphi(u_1, u_2)$ to $\varphi(u_2, u_1)$ and t to t^{-1} if necessary, we can assume $i < j$. We consider three cases:

Case 1 $j < n$.

By the induction hypothesis, there is a term s' such that

$$(\forall x_1, \ldots, x_{n-1}) \, (s'(x_1, \ldots, x_{n-1}) = \{(u_1, \ldots, u_{n-1}) : \\ u_1 \in x_1, \ldots, u_{n-1} \in x_{n-1} \wedge \varphi(u_i, u_j)\}) \, .$$

Then the rudimentary term $s(v_1, \ldots, v_n) = s'(v_1, \ldots, v_{n-1}) \times v_n$ has the desired property.

Case 2 $j = n$ and $i < n - 1$.

By the previous case, there is a term s' such that

$$(\forall x_1, \ldots, x_n) \, (s'(x_1, \ldots, x_n) = \{(u_1, \ldots, u_n) : \\ u_1 \in x_1, \ldots, u_n \in x_n \wedge \varphi(u_i, u_{n-1})\}) \, .$$

Then the rudimentary term $s(v_1, \ldots, v_n) = R_8(s'(v_1, \ldots, v_n, v_{n-1})))$ has the desired property.

Case 3 $j = n$ and $i = n - 1$.

Then the rudimentary term $s(v_1, v_2, \ldots, v_{n-1}, v_n) \equiv R_9(t(v_{n-1}, v_n) \times (v_1 \times \ldots \times v_{n-2}))$ has the desired property.

This completes the proof of the claim.

Let's call the formula φ simplified if it does not contain any of the symbols $\wedge$, $\forall$, $=$, nor the subformula $v_i \in v_i$ for certain i. First we prove the lemma for all simplified φ that do not contain any quantifiers. This is done by induction on the simplified formula φ.

Case 1 φ is atomic.

Using the claim above and the assumption that φ is simplified, it suffices to consider here the case $\varphi(v_1, v_2) \equiv v_1 \in v_2$.
In this case, $t(v_1, v_2) \equiv R_5(v_1, v_2)$ has the desired property.

Case 2 $\varphi \equiv \neg\varphi_1(v_1, \ldots, v_n)$.

By the induction hypothesis, there is a rudimentary term $s(v_1, \ldots, v_n)$ such that

$$(\forall x_1, \ldots, x_n) \, (s(x_1, \ldots, x_n) = \{(u_1, \ldots, u_n) : \\ u_1 \in x_1, \ldots, u_n \in x_n \wedge \varphi_1(u_1, \ldots, u_n)\}.)$$

In this case, $t(v_1, \ldots, v_n) \equiv v_1 \times \ldots \times v_n \setminus s(v_1, \ldots, v_n)$ has the desired property.

Case 3 $\varphi \equiv \varphi_1(v_1, \ldots, v_n) \vee \varphi_2(v_1, \ldots, v_n)$.

By the induction hypothesis, there are rudimentary terms $s_i(v_1, \ldots, v_n)$ such that

$$(\forall x_1, \ldots, x_n) \, (s_i(x_1, \ldots, x_n) = \{(u_1, \ldots, u_n) : \\ u_1 \in x_1, \ldots, u_n \in x_n \wedge \varphi_i(u_1, \ldots, u_n)\}) \, .$$

Then the rudimentary term $t(v_1, \ldots, v_n) \equiv \bigcup\{s_1(v_1, \ldots, v_n), s_2(v_1, \ldots, v_n)\}$ has the desired property.

This completes the proof in case φ is simplified and does not contain any quantifiers. Next, we work with induction on the number of quantifiers in φ.
We consider $\varphi(v_1, \ldots, v_n) \equiv (\exists v_{n+1} \in v_i)(\varphi_1(v_1, \ldots, v_n, v_{n+1}))$.
Consider now the simplified Δ_0-formula

$$\psi(v_1, \ldots, v_n, v_{n+1}) \equiv \varphi_1(v_1, \ldots, v_{n+1}) \wedge v_{n+1} \in v_i.$$

Since ψ has less occurences of the symbol $\exists$ than φ, the induction hypothesis gives a term $s(v_1, \ldots, v_{n+1})$ such that

$$(\forall x_1, \ldots, x_{n+1})\,(s(x_1, \ldots, x_{n+1}) = \{(u_1, \ldots, u_{n+1}) : \\ u_1 \in x_1, \ldots, u_n \in x_n, u_{n+1} \in x_{n+1} \wedge \psi(u_1, \ldots, u_{n+1})\})\,.$$

Then the rudimentary term $t(v_1, \ldots, v_n) \equiv \mathrm{dom}(s(v_1, \ldots, v_n, \cup v_i))$ has the desired property.

To complete the proof, we only need to find for every Δ_0-formula $\varphi(v_1, \ldots, v_n)$ a simplified Δ_0-formula $\psi(v_1, \ldots, v_n)$ such that

$$\mathsf{Extensionality} \vdash \forall x_1, \ldots, x_n \quad \varphi(x_1, \ldots, x_n) \iff \psi(x_1, \ldots, x_n).$$

We construct ψ by first replacing every occurence in φ of $v_i \in v_i$ by $(\exists v_k \in v_i)(v_k = v_i)$ and then replacing every occurence in φ of $v_i = v_j$ by

$$(\forall v_k \in v_i)(v_k \in v_j) \wedge (\forall v_k \in v_j)(v_k \in v_i).$$

Finaly, we rewrite the formula thus obtained to express the occurences of $\forall$ and $\wedge$ in terms of $\exists$, $\vee$ and $\neg$.

□

Corollary 1.1

$$\mathsf{Extensionality} + \mathsf{Rudimentary\ Closure} \vdash \Delta_0\text{-}\mathsf{separation}$$

Proof By combining Lemma 1.1 and Theorem 1.1. □

Definition 1.3
The theory DB_0 consists of the following axioms.

- Axiom of Extensionality
- Axiom of Rudimentary Closure
- Axiomscheme of Δ_0-Separation

The notation DB_0 is taken from Mathias and Bowler (2012) and is short for "Devlin Basic", a reference to the book Devlin (1984).

Corollary 1.2
DB_0 is finitely axiomatisable.

Proof
By Theorem 1.1, we can simply delete the axiomscheme of Δ_0-separation from DB_0. □

Regularity

The axiom of (set) regularity is formulated in the usual way.

Axiom of Regularity

$$(\forall x \neq \emptyset)(\exists y)(y \in x \wedge \neg(\exists z)((z \in x) \wedge (z \in y))).$$

As we know, over the axioms $\mathsf{ZF} \setminus \{\mathsf{Regularity}\}$, the axiom of regularity is equivalent to each of the following two schemes:

Class Regularity Scheme

This scheme asserts that every non-empty class A has an $\in$-minimal element.

$\in$-Induction Scheme

This scheme asserts for every formula $\varphi(x)$ that

$$((\forall x)((\forall y \in x \varphi(y)) \Rightarrow \varphi(x))) \Rightarrow (\forall x \varphi(x)).$$

One should be a little careful for the following.

Caveat: Without the full strength of the separation and replacement schemes, set regularity is weaker than class regularity and also weaker than the $\in$-induction scheme.

Also transitive closures and the set theoretical rank of sets are not well-defined in a general DB_0-model, so we need to formulate their existence in separate axioms.

Axiom of Transitive Closure

$$(\forall x)(\exists y)(y = \mathrm{Trcl}(x))$$

Axiom of Rank

$$(\forall x)(\exists y)(y = \mathrm{Rank}(x))$$

We can then properly adjust the axiom schemes above to make up for the weakened strength of Δ_0-separation.

Δ_0-class Regularity Scheme

This scheme asserts that every non-empty Δ_0-class A has an $\in$-minimal element.

$\in$-Induction Scheme

This scheme asserts for every Δ_0-formula $\varphi(x)$ that

$$((\forall x)((\forall y \in x\varphi(y)) \Rightarrow \varphi(x))) \Rightarrow (\forall x\varphi(x)).$$

Theorem 1.2
With respect to the theory $\mathsf{DB}_0 + \mathsf{Transitive\ closure}$ the following three are equivalent:

(1) Regularity

(2) Δ_0-class regularity

(3) Δ_0-$\in$-induction.

Proof

(1) $\Rightarrow$ (2)

Let A be a non-empty Δ_0-class, $x \in A$. If $x \cap A = \emptyset$, then x is $\in$-minimal in A. Else, by Transitive closure, choose t transitive with $x \subseteq t$. By Δ_0-separation, $\emptyset \neq t \cap A \in V$, by Regularity choose y $\in$-minimal in $A \cap t$. Suppose y is not $\in$-minimal in A, then there is a $z \in y \cap A$, but then $z \in y \in t \Rightarrow z \in t$, hence $z \in t \cap y \cap A$: a contradiction.

(2) $\Rightarrow$ (3)

Suppose $(\forall x)((\forall y \in x\varphi(y)) \Rightarrow \varphi(x))$, where φ is a Δ_0-formula. Consider the Δ_0-class $A = \{x : \neg\varphi(x)\}$. By Δ_0-class regularity, if A should be non-empty, it should possess an $\in$-minimal element x, but then the assumption implies $\varphi(x)$, which is in contradiction with $x \in A$. It follows that A is empty.

(3) $\Rightarrow$ (1)

Let z be a non-empty set. Consider the Δ_0-formula $\varphi(x) \equiv x \notin z$. By the contraposition of $((\forall x)((\forall y \in x\varphi(y)) \Rightarrow \varphi(x))) \Rightarrow (\forall x\varphi(x))$, we find $(\exists x)((\forall y \in x\varphi(y)) \wedge \neg\varphi(x)))$. But this last condition is equivalent to the existence of an $\in$-minimal element of z. □

1.3 Primitive Recursive Set functions

In 1971, Ronald B. Jensen and Carol Karp introduced a hierarchy of Primitive Recursive Set functions, consisting of definable set functions $F : V^k \to V$ (Jensen and Karp (1971)). These generalise the more familiar arithmetic Primitive Recursive functions in the sense that we can re-obtain the arithmetical variants by restricting all Primitive Recursive Set functions $F : V^k \to V$ to the set of hereditarily finite sets H_ω.

Definition 1.4
Starting from a collection $C = \{\varphi_1(v_1, w), \ldots, \varphi_k(v_1, w)\}$ of formulas, we inductively define the C-PRS formulas as follows.

Base functions:

- $\varphi_1(v_1, w), \ldots, \varphi_k(v_1, w)$ are C-PRS formulas,
- $\varphi_{\mathrm{proj},n,i}(v_1, \ldots, v_n, w) \equiv w = v_i$ is a C-PRS formula for any $n < \omega$ and $1 \leq i \leq n$,
- $\varphi_{\mathrm{const}}(v_1, w) \equiv w = \emptyset$ is a C-PRS formula,
- $\varphi_{\mathrm{S}}(v_1, v_2, w) \equiv w = v_1 \cup \{v_2\}$ is a C-PRS formula,
- $\varphi_{\mathrm{test}}(v_1, v_2, v_3, v_4, w) \equiv (v_1 \in v_2 \Rightarrow w = v_3) \wedge (v_1 \notin v_2 \Rightarrow w = v_4)$ is a C-PRS formula.

Generating rules:

- If $\varphi_1(v_1, \ldots, v_{n-1}, w)$ and $\varphi_2(v_1, \ldots, v_{n-1}, v_n, \ldots, v_{n+k}, w)$ are C-PRS formulas, then

  $$\psi(v_1, \ldots, v_{n-1}, \ldots, v_{n+k}, w) \equiv (\exists v)\,(\varphi_2(v_1, \ldots, v_{n-1}, v, \ldots, v_{n+k}, w) \\ \wedge \varphi_1(v_1, \ldots, v_{n-1}, v))$$

  is a C-PRS formula.
- If $\varphi(v_0, v_1, \ldots, v_k, w)$ is a C-PRS formula, then

  $$\psi(v_1, \ldots, v_k, w) \equiv (\exists v)\mathrm{fun}(v) \wedge \mathrm{dom}(v) \supseteq v_1 \wedge \mathrm{trans}(\mathrm{dom}(v)) \\ \wedge\, \varphi(\bigcup v[v_1], v_1, \ldots, v_k, w) \wedge (\forall v' \in \mathrm{dom}(v))\varphi(\bigcup v[v'], v', \ldots, v_k, v(v'))$$

  is a C-PRS formula.

When $C = \emptyset$, we will use the terminology "PRS formula" instead of "C-PRS formula".

Definition 1.5
A relation $F \subseteq V^k \times V$ is a C-Primitive Recursive Set function if there exists a C-PRS formula $\varphi(v_1, \ldots, v_k, w)$ such that

$$F = \{((x_1, \ldots, x_k), y) : \varphi(x_1, \ldots, x_k, y)\}.$$

A Primitive Recursive Set function is by definition the same as an $\emptyset$-Primitive Recursive Set function.

Lemma 1.3
For any Primitive Recursive Set function F, the theory $\mathsf{DB}_0 + \mathsf{Regularity}$ proves that F is a function.

Proof
We have to prove, for any PRS-formula $\varphi(v_1, \ldots, v_k, w)$, that:

$$\forall x_1, \ldots, x_k \forall y_1, y_2 (\varphi(x_1, \ldots, x_k, y_1) \wedge \varphi(x_1, \ldots, x_k, y_2) \Rightarrow y_1 = y_2)$$

It is a trivial matter to check this for the base functions, we proceed by induction on the PRS-formula φ.

Substitution rule:
Since $\varphi(x_1,\ldots,x_k,y_1)$, there exists z_1 with

$$\varphi_2(x_1,\ldots,x_{n-1},z_1,\ldots,x_{n+k},y_1)\wedge\varphi_1(x_1,\ldots,x_{n-1},z_1).$$

Since $\varphi(x_1,\ldots,x_k,y_2)$, there exists z_2 with

$$\varphi_2(x_1,\ldots,x_{n-1},z_2,\ldots,x_{n+k},y_2)\wedge\varphi_1(x_1,\ldots,x_{n-1},z_2).$$

Applying the induction hypothesis on φ_1 gives $z_1=z_2$. Applying the induction hypothesis on φ_2 then gives $y_1=y_2$.

Recursion rule:
Since $\varphi(x_1,\ldots,x_k,y_1)$, there exists z_1 with

$$\begin{aligned}\mathrm{fun}(z_1)\wedge\mathrm{dom}(z_1)\supseteq x_1\wedge\mathrm{trans}(\mathrm{dom}(z_1))\wedge\varphi(\textstyle\bigcup z_1[x_1],x_1,\ldots,x_k,y_1)\\ \wedge(\forall v'\in\mathrm{dom}(z_1))\varphi(\textstyle\bigcup z_1[v'],v',\ldots,x_k,z_1(v')).\end{aligned}$$

Since $\varphi(x_1,\ldots,x_k,y_2)$, there exists z_2 with

$$\begin{aligned}\mathrm{fun}(z_2)\wedge\mathrm{dom}(z_2)\supseteq x_1\wedge\mathrm{trans}(\mathrm{dom}(z_2))\wedge\varphi(\textstyle\bigcup z_2[x_1],x_1,\ldots,x_k,y_2)\\ \wedge(\forall v'\in\mathrm{dom}(z_2))\varphi(\textstyle\bigcup z_2[v'],v',\ldots,x_k,z_2(v')).\end{aligned}$$

We now use Δ_0-induction on the formula

$$\tau(x)\equiv x\in\mathrm{dom}(z_1)\cap\mathrm{dom}(z_2)\Rightarrow z_1(x)=z_2(x).$$

Suppose that $(\forall y\in x)\tau(y)$, we will then prove $\tau(x)$.
If $x\in\mathrm{dom}(z_1)\cap\mathrm{dom}(z_2)$, we have that

$$\varphi(\textstyle\bigcup z_1[x],x,\ldots,x_k,z_1(x))$$

and

$$\varphi(\textstyle\bigcup z_2[x],x,\ldots,x_k,z_2(x)).$$

Suppose $y\in x$. Because $\mathrm{trans}(\mathrm{dom}(z_1))$ and $\mathrm{trans}(\mathrm{dom}(z_2))$, it follows that $y\in\mathrm{dom}(z_1)\cap\mathrm{dom}(z_2)$, we can apply the Δ_0-induction hypothesis to obtain $z_1(y)=z_2(y)$. This proves that $\bigcup z_1[x]=\bigcup z_2[x]$.
Finally, applying the induction hypothesis on φ leads to $z_1(x)=z_2(x)$.
Combining

$$\varphi(\textstyle\bigcup z_1[x_1],x_1,\ldots,x_k,y_1)$$

and

$$\varphi(\textstyle\bigcup z_2[x_1],x_1,\ldots,x_k,y_2)$$

then gives $y_1=y_2$. □

Axiomscheme of Primitive Recursive Closure

This scheme asserts for every Primitive Recursive Set function $F(x_1,\ldots,x_k)$:

$$(\forall x_1,\ldots,x_k)(\exists y)\quad y=F(x_1,\ldots,x_k).$$

Equivalently, this scheme asserts for any k-ary Primitive Recursive Set function $F(x_1, \ldots, x_k)$ that $\mathrm{dom}(F) = V^k$.

Checking that a particular function $F : V^k \to V$ is a Primitive Recursive Set function is entirely analogous to checking that an arithmetical function is primitive recursive (see the course *Berekenbaarheid en Complexiteit* (Brinkmann (2016))), and we refer to Jensen and Karp (1971) for the necessary checks underlying the following three lemma's.

Lemma 1.4
If $t(v_1, \ldots, v_k)$ is a rudimentary term, then

$$F(x_1, \ldots, x_k)\ t(x_1, \ldots, x_k)$$

is a Primitive Recursive Set function.

Lemma 1.5
The following functions are Primitive Recursive Set functions.

- $(\alpha, \beta) \mapsto \alpha + \beta$,
- $(\alpha, \beta) \mapsto \alpha.\beta$,
- $x \mapsto \mathrm{Trcl}(x)$,
- $x \mapsto \mathrm{Rank}(x)$.

Corollary 1.3
$\mathsf{DB}_0 + \mathsf{Regularity} + \mathsf{Primitive\ Recursive\ Closure} \vdash \mathsf{Transitive\ Closure} + \mathsf{Rank}$

Lemma 1.6
If F is a Primitive Recursive Set function, then

$$(\forall x)(F[x] \in V).$$

Collapse

Definition 1.6
A relation R on a domain D is set-like if

$$(\forall x \in D) \quad \{u \in D : uRx\} \in V.$$

A relation R on a domain D is well-founded if it is set-like and any non-empty set $x \subseteq D$ has an R-minimal element.
A relation R on a domain D is a well-order if it is a well-founded total order.
A relation R on a domain D is extensional if

$$(\forall x_1, x_2 \in D)\left[(\forall y)(y \in x_1 \iff y \in x_2) \Rightarrow x_1 = x_2\right].$$

Even over the weak set theory DB_0, well-founded relations give rise to induction principles.

Lemma 1.7 (DB_0)
Let R be a well-founded relation on the set u and let φ be a Δ_0-formula. Then

$$[(\forall x \in u)((\forall y R x \varphi(y)) \Rightarrow \varphi(x))] \Rightarrow \forall x \in u\ \varphi(x).$$

Proof
Suppose $(\forall x \in u)((\forall y R x \varphi(y)) \Rightarrow \varphi(x))$.
Consider the set $s = \{x : x \in u \wedge \neg\varphi(x)\}$. If s is non-empty, we can choose z minimal in s. But then the assumption implies $\varphi(z)$, which is in contradiction with $z \in s$. □

Definition 1.7
Let R be a well-founded relation on a domain D, a function $f : D \to V$ is called a (Mostowski) *collapse* (function) for R if

$$(\forall x \in D) \quad f(x) = \{f(y) : yRx\}.$$

Axiom of Collapse

If $r \in V$ is a well-founded relation on the set x, there exists $f \in V$ such that $f : x \to V$ is a collapse for r.

In ZF set theory, the statement in this axiom will of course follow from the Mostowski collapse theorem.

1.4 Simpson's system of weak set theory

Definition 1.8
The theory Sim^- consists of the following axioms.

1. Axiom of Extensionality
2. Axiomscheme of Δ_0-separation
3. Axiomscheme of primitive recursive closure
4. Axiom of Infinity
5. Axiom of Regularity.

The theory Sim consists of Sim^- together with the axiom of collapse.

Lemma 1.8 (Sim^-)
Let R be a well-founded relation on the set d and let $f : d \to V$ be a collapse for R. Suppose moreover that f is a Δ_0-function. Then:

1. If $g : d \to V$ is a second collapse for R and g is also a Δ_0-functions, then $f = g$.
2. $f[d]$ is a transitive class .
3. If R is extensional, then f is injective.
4. If R is extensional, $(\forall x, y)(xRy \iff f(x) \in f(y))$.
5. If R is a well-order, then $f[d] \in \mathrm{On}$.

Proof

1. By Δ_0-induction for the well-founded relation R on the Δ_0-formula
$$\varphi(x) \equiv f(x) = g(x).$$
2. Let $x \in d$, we have to prove that $f(x) \subseteq f[d]$, but this is clear since $f(x) = \{f(y) : yRx\} \subseteq f[d]$.
3. By Δ_0-$\in$-induction on the Δ_0-formula
$$\varphi(x) \equiv (x \in f[d] \Rightarrow (\exists! v \in d) f(v) = x).$$
4. $\boxed{\Rightarrow}$ if xRy then clearly $f(x) \in \{f(u) : uRy\} = f(y)$.
$\boxed{\Leftarrow}$ if $f(x) \in f(y) = \{f(u) : uRy\}$, then $f(x) = f(u)$ for certain uRy, but by injectivity of f, it follows that $x = u$, hence xRy.
5. f is an isomorphism between (d, R) and $(f[d], \in)$. Hence, if R is total order on d, $\in$ is a total order on $f[d]$.

□

Lemma 1.9 (Sim)
Any two well-orderings are comparable.

Proof
Let $W_1, <$ and $W_2, <$ be well-orderings. By the axiom of collapse, there exists for both orders a collapsing function f_i. By Lemma 1.8, these collapse functions are actually order-isomorphisms $f_i : W_i \to \alpha_i$, for certain ordinals α. The result follows since any two ordinals are comparable by the $\subseteq$-relation. □

Countability

Sim does not prove the existence of a set that is not (hereditarily) countable. Indeed, let's define:
$$\mathsf{Countability} \iff V = H_{\omega_1}.$$
Note that we should be a little careful in the definition of H_{ω_1}, since we did not incorporate the axiom of choice in the theory Sim. We will work with the following definition:
$$H_{\omega_1} = \{x : \mathrm{Count}(\mathrm{Trcl}(x))\}.$$
Here $\mathrm{Count}(x)$ is an $\mathcal{L}_\in$-formula formalising that x is countable in the precise sense that there is an injection from x to ω.

Then the following holds.

Lemma 1.10
$$\mathsf{Sim} \vdash (\mathsf{Sim} + \mathsf{Countability})^{H_{\omega_1}}$$

Proof
Let $\mathcal{M} \models \mathsf{Sim}$, we work in $\mathcal{M}$.

Extensionality, Regularity and $x = \omega$ can be formalized by Δ_0^{Sim}-formulas, so it follows that for any transitive class T, containing ω,

$$T \models \mathsf{Extensionality} + \mathsf{Regularity} + \mathsf{Infinity}$$

holds.
To check Δ_0-separation, choose an arbitrary Δ_0-formula φ, with y not free in φ. The formula $\varphi^{H_{\omega_1}}$ is again Δ_0. Let $x \in H_{\omega_1}$. By Δ_0-separation, there is y such that $y = \{z : (z \in x) \wedge \varphi^{H_{\omega_1}}\}$.
Because H_{ω_1} is closed under subsets, we have $y \in H_{\omega_1}$. Since

$$H_{\omega_1} \models y \models \{z : (z \in x) \wedge \varphi\},$$

we have verified that $H_{\omega_1} \models \Delta_0$-separation.

For primitive recursive closure, one repeats the induction performed in Lemma 1.3 to check for all PRS-formulas $\varphi(v_1, \ldots, v_k, w)$

$$(\forall x_1, \ldots, x_k \in H_{\omega_1})(\forall y)(\varphi(x_1, \ldots, x_k, y) \Rightarrow y \in H_{\omega_1}).$$

Since every PRS-formula is Σ_1, it follows that

$$H_{\omega_1} \models \mathsf{Primitive\ Recursive\ Closure}.$$

To check collapse in H_{ω_1}, let r be a well-founded relation on the set x. Since x is countable, so is $f[x]$. Since $f[x]$ is transitive, $f[x] \in H_{\omega_1}$.
It follows that $f \subseteq x \times f[x] \in H_{\omega_1}$. Since the statement that f is the collapse of r is Δ_0, $H_{\omega_1} \models \mathsf{Collapse}$.

□

Lemma 1.11

$$\mathsf{ZF} \vdash \mathsf{Sim}$$

Proof
The only non-trivial step entails checking that

$$\mathsf{ZF} \vdash \mathsf{Collapse},$$

but this follows straight from the Mostowski-Collapse Theorem. □

Combining the two foregoing lemma's we deduce that $(H_{\omega_1}, \in)$ is a model of the set theory Sim that does not believe the existence of uncountable sets.

Corollary 1.4 (ZF)

$$(H_{\omega_1}, \in) \models \mathsf{Sim} + \mathsf{Countability}$$

1.5 Developing mathematics in Sim

In any model of Sim we find the following model of PA:

$$\mathcal{N} = (\omega, 0, 1, S^{\mathcal{N}} : \omega \to \omega : \alpha \mapsto \alpha \cup \{\alpha\}, +^{\mathcal{N}} : \omega \times \omega \to \omega : (\alpha, \beta) \mapsto \alpha + \beta,$$
$$\cdot^{\mathcal{N}} : \omega \times \omega \to \omega : (\alpha, \beta) \mapsto \alpha.\beta, <^{\mathcal{N}} = \in \restriction \omega)$$

However, not every model of PA can be obtained in this way. Indeed let $\mathcal{L}_2$ be the language of second order arithmetic, let $\mathcal{M}$ be the $\mathcal{L}_2$-structure $(\mathcal{N}, \mathcal{S}^{\mathcal{M}} = P(\omega))$, then $\mathcal{M}$ satisfies the subsystem of second order arithmetic ATR_0 (Arithmetical Transfinite Recursion).

Theorem 1.3 (Sim)

$$\mathcal{M} \models \mathsf{ATR}_0$$

Proof
In addition to ACA_0, the model of second order arithmetic $\mathcal{M}$ satisfies the principle of comparability of well-orderings (because of Lemma 1.9), which is equivalent to ATR_0 over ACA_0, see Simpson (2009). □

As is common mathematical practice, we can denote the above structures $\mathcal{N}$ and $\mathcal{M}$, by $\mathbb{N}$. Since $\mathbb{N}$ satisfies the cancellation laws for the operation $+$, there is a canonical way for embedding $\mathbb{N}$ in an ordered ring $\mathbb{Z}$. Next, since $\mathbb{Z}$ is a domain, there is a canonical way to embed it in its ordered fractionfield $\mathbb{Q}$. Both $\mathbb{Z}$ and $\mathbb{Q}$ can be proved sets in Sim^-.

Definition 1.9
A *real number* is a non-empty subset $r \subseteq \mathbb{Q}$ that is bounded from above, has no maximum and satisfies

$$(\forall p \in r)(\forall q \in \mathbb{Q})(q < p \Rightarrow q \in r).$$

$\mathbb{R}$ is the Δ_0-class $\{r : r \text{ is a real number}\}$.

We define the following order on the reals.

Definition 1.10

$$(\forall r_1, r_2 \in \mathbb{R})(r_1 \leq r_2 \iff r_1 \subseteq r_2).$$

And we can then retrieve the supremumprinciple.

Lemma 1.12 (supremumprinciple; Sim^-)
Suppose $A \subseteq \mathbb{R}$ is bounded from above and suppose $\bigcup A \in V$, then the class of reals A has a supremum in $\mathbb{R}$. In particular, every set of reals that is bounded from above has a supremum.

Proof
Because the class of reals A is bounded from above, the set $\bigcup A$ is a non-empry bounded subset of $\mathbb{Q}$. Since A consists of reals, $\bigcup A$ is a real again. Since the order on $\mathbb{R}$ is given by inclusion, $\bigcup A$ is also the smallest upperbound for A. Because the operation $x \mapsto \bigcup A$ is primitive recursive (even rudimentary), the second claim follows from the first. □

Unsurprisingly, we have:

Lemma 1.13 (Sim^-)
The class $\mathbb{R}$ is not countable.

Proof
By an easy diagonal argument, we can construct for any $f : \omega \to \mathbb{R}$ a real $r_f \in \mathbb{R}$ not in the image of f. See Simpson (2009) for details. □

Corollary 1.5
$\mathsf{Sim}^- \nvdash \mathbb{R} \in V$.

Definition 1.11

$$r_1 + r_2 := \{q_1 + q_2 : q_1 \in r_1, q_2 \in r_2\}.$$

$$r_1.r_2 := \begin{cases} 0 & r_1 = 0 \vee r_2 = 0 \\ \{q_1.q_2 : q_1 \in r_1 \cap \mathbb{Q}_{>0}, q_2 \in r_2\} & r_1 > 0 \wedge r_2 > 0 \\ (-r_1).r_2 & r_1 < 0 \wedge r_2 > 0 \\ r_1.(-r_2) & r_1 > 0 \wedge r_2 < 0 \\ (-r_1).(-r_2) & r_1 < 0 \wedge r_2 < 0. \end{cases}$$

From here, one can go on to define the euclidean metric $d(x, y) = |x - y|$ on $\mathbb{R}$ and from there the notion of open and closed subclasses of $\mathbb{R}$.

References and Remarks

The rudimentary set functions are commonly used in studies of the constructible universe such as fine structure theory, see for example Jech (2006). The proof of Theorem 1.1 given here is based on the corresponding proof in Jech (2006). The proof of Lemma 1.8 is based on an analogue proof in the course notes Koepke (2016a). The theory Sim was introduced in Simpson (1982), where the relation with ATR_0 is established. Because of the relation between the theory Sim + Countability and ATR_0, the theory Sim (and variations of this theory), is also denoted by $\mathsf{ATR}_0^{\mathsf{Set}}$. Our short exposition of the development of mathematics in Sim owes to Weaver (2005) and Koepke (2016a).

We think it might be of interest to further investigate the exact relationships between various weak set theories and their models. We think many questions here have remained unasked, perhaps because for many set theorists, set theory starts at the level of ZF. An interesting work where this proposed theme has been considered is Mathias (2006), where various surprising models of weak set theories are constructed. Also, it might be interesting to try to determine, starting from a weak base set theory (such as DB_0+ Regularity), exactly what it takes extra to obtain certain results from ZF-set theory, perhaps in the style of Simpson's program of reverse mathematics.

2

Γ-trees

2.1 Γ-trees

In this section, by *tree*, we will mean *finite tree*.

Definition 2.1
A Γ*-tree* $\tau = (V, E, f, r)$ is a rooted tree (V, E, r) with root r, together with a partial function $f : E \to \{0, 1\}$, such that

$$\mathrm{dom}(f) = \{e \in E : e \text{ not adjacent to } r\}.$$

Hence, a Γ-tree is a rooted tree together with a $\{0, 1\}$-colouring of the edges that are not adjacent to its root. We will think about this $\{0, 1\}$-colouring as giving a direction to every coloured edge of τ. It is therefore equivalent to define a Γ-tree as a rooted tree whose edges adjacent to its root are undirected, and whose edges not adjacent to the root are directed.

Definition 2.2
Two Γ-trees $\tau_1 = (V_1, E_1, f_1, r_1)$ and $\tau_2 = (V_2, E_2, f_2, r_2)$ are *isomorphic* when there exists a bijection $b : V_1 \to V_2$ with the following properties:

$$\forall v, w \in V_1 \quad (v, w) \in E_1 \Leftrightarrow (b(v), b(w)) \in E_2,$$

$$b(r_1) = r_2,$$

$$\forall v, w \in V_1 \quad f_2((b(v), b(w))) = f_1((v, w)).$$

Definition 2.3
We fix a set Γ_{tree} containing one element of each isomorphism class of Γ-trees.

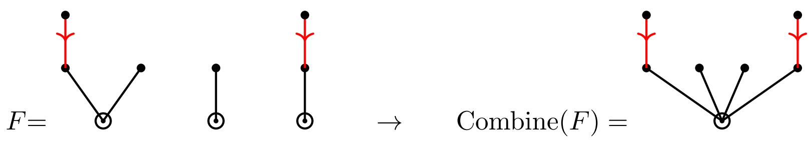
Definition 2.4
A rooted tree is *planted* when the degree of its root equals 1.

Definition 2.5
The *order* $|\tau|$ of a tree τ is defined as the number of vertices (this includes the root) of τ.

Definition 2.6
The *height* $\mathrm{ht}(\tau)$ of a Γ-tree τ is defined as the height of the corresponding rooted tree.

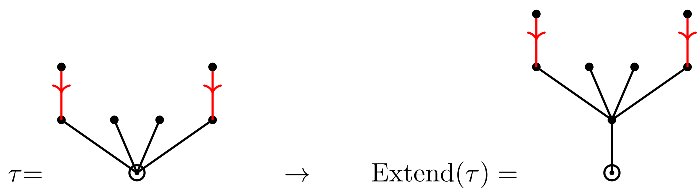
Definition 2.7
Given a forest $F = \{\tau_1, \tau_2, \ldots, \tau_k\}_{\text{multi}}$ of rooted trees, we define $\text{Combine}(F)$ as the rooted tree obtained by identifying the roots of $\tau_1, \tau_2, \ldots, \tau_k$ (here the brackets $\{\ \}_{\text{multi}}$ indicate a multiset).
When the trees in F are (partially) coloured, this induces a (partial) colouring on $\text{Combine}(F)$ (see Figure 2.1).

Figure 2.1: Example Combine



Definition 2.8
Given a $\{0, 1\}$-coloured tree τ with root r, we define $\text{Extend}(\tau)$ as the Γ-tree obtained by adding to τ a new root r', adding the edge $\{r, r'\}$, and colouring determined by the colouring on τ (see Figure 2.2).

Figure 2.2: Example Extend

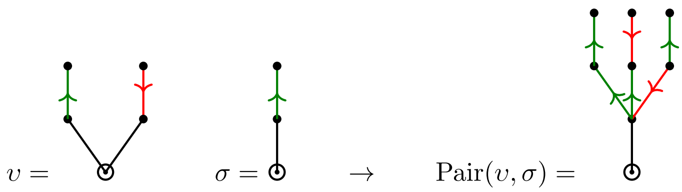


Definition 2.9
Given two Γ-trees σ and υ, we define $\tau = \text{Pair}(\sigma, \upsilon)$ as the Γ-tree obtained as follows.

- Construct $\text{Extend}(\text{Combine}(\sigma, \upsilon))$.
- Every edge of σ that was adjacent to the root of σ gets colour 1.
- Every edge of υ that was adjacent to the root of υ gets colour 0.

(See Figure 2.3)

Figure 2.3: Example Pair

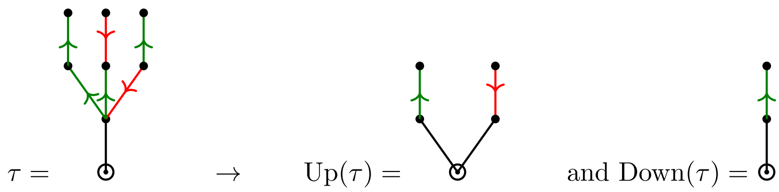


Definition 2.10
Given a planted Γ-tree τ, we define Γ-trees $\mathrm{Up}(\tau)$ and $\mathrm{Down}(\tau)$ as the unique Γ-trees satisfying

$$\tau = \mathrm{Pair}(\mathrm{Up}(\tau), \mathrm{Down}(\tau)).$$

(See Figure 2.4.)

Figure 2.4: Example $\mathrm{Up}(\tau)$, $\mathrm{Down}(\tau)$



2.2 Ordering Γ-trees

Curiously, the class of Γ-trees admits an interesting ordering, which we define by recursion on the height of Γ-trees.

Definition 2.11

- If $0 = \mathrm{ht}(\tau_1) < \mathrm{ht}(\tau_2)$, then $\tau_1 < \tau_2$.
- Let τ_1 and τ_2 be two Γ-trees, both planted.
  Then

$$\tau_1 < \tau_2 \iff \begin{cases} & (\mathrm{Up}(\tau_1) < \mathrm{Up}(\tau_2) \quad \wedge \quad \mathrm{Down}(\tau_1) < \tau_2) \\ \text{or} & (\mathrm{Up}(\tau_1) \cong \mathrm{Up}(\tau_2) \quad \wedge \quad \mathrm{Down}(\tau_1) < \mathrm{Down}(\tau_2)) \\ \text{or} & (\mathrm{Up}(\tau_1) > \mathrm{Up}(\tau_2) \quad \wedge \quad \tau_1 \leq \mathrm{Down}(\tau_2)). \end{cases}$$

- Let $F_1 = \{\tau_1, \ldots, \tau_k\}_{\mathrm{multi}}$ and $F_2 = \{\sigma_1, \ldots, \sigma_l\}_{\mathrm{multi}}$ be two forests of planted Γ-trees.
  We abbreviate $\tau_1 < \tau_2 \vee \tau_1 \cong \tau_2$ by $\tau_1 \leq \tau_2$.

We define $F_1 < F_2$ to hold if there exist $\pi \in \mathrm{Sym}_k$ and $\rho \in \mathrm{Sym}_l$, such that
$\tau_{\pi(1)} \geq \tau_{\pi(2)} \geq \ldots \geq \tau_{\pi(k)}$ and $\sigma_{\rho(1)} \geq \sigma_{\rho(2)} \geq \ldots \geq \sigma_{\rho(l)}$ and either

$$k < l \qquad \text{and} \qquad (\forall\ 1 \leq i \leq k)(\tau_{\pi(i)} \cong \sigma_{\rho(i)})$$

or

$$(\exists\ 1 \leq j \leq \min(k,l)) \text{ such that } \tau_{\pi(j)} < \sigma_{\rho(j)} \text{ and } (\forall\ 1 \leq i < j)(\tau_{\pi(i)} \cong \sigma_{\rho(i)}).$$

- Lastly, we extend our definition to all Γ-trees by stipulating

$$\mathrm{Combine}(F_1) < \mathrm{Combine}(F_2) \iff F_1 < F_2.$$

We now check that $<$ defines an order relation on the class of Γ-trees which is total on the isomorphism classes.

Theorem 2.1 (Sim)
For every two non-isomorphic Γ-trees τ and σ either $\tau < \sigma$ or $\sigma < \tau$.

Proof
By induction on $|\tau| + |\sigma|$.

First assume that τ and σ are planted.
Suppose $\neg(\tau < \sigma)$.
By the induction hypothesis the pairs $(\mathrm{Up}(\tau), \mathrm{Up}(\sigma))$, $(\mathrm{Down}(\tau), \sigma)$, $(\mathrm{Down}(\sigma), \tau)$ and $(\mathrm{Down}(\tau), \mathrm{Down}(\sigma))$ are all comparable.

Case 1 $\mathrm{Up}(\tau) < \mathrm{Up}(\sigma)$.

It then follows that $\mathrm{Down}(\tau) \geq \sigma$, but this implies that $\tau > \sigma$.

Case 2 $\mathrm{Up}(\tau) \cong \mathrm{Up}(\sigma)$.

It now follows that $\mathrm{Down}(\tau) \geq \mathrm{Down}(\sigma)$.

Case 2.1 $\mathrm{Down}(\tau) \cong \mathrm{Down}(\sigma)$.

This implies that $\tau \cong \sigma$, a contradiction.

Case 2.2 $\mathrm{Down}(\tau) > \mathrm{Down}(\sigma)$.

It now follows that $\tau > \sigma$.

Case 3 $\mathrm{Up}(\tau) > \mathrm{Up}(\sigma)$.

It follows that $\tau > \mathrm{Down}(\sigma)$, which implies $\tau > \sigma$.

Now suppose that we have trees $\tau = \mathrm{Combine}(F_1), \sigma = \mathrm{Combine}(F_2)$, where $F_1 = \{\tau_1, \ldots, \tau_k\}_{\mathrm{multi}}$ and $F_2 = \{\sigma_1, \ldots, \sigma_l\}_{\mathrm{multi}}$ are two forests of planted Γ-trees. Without loss of generality (using the induction hypothesis), assume that $\tau_1 \geq \tau_2 \geq \ldots \geq \tau_k$ and $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_l$.

Case 1 $(\exists j \leq \min(k,l))\ \neg(\tau_j \cong \sigma_j)$.

Choose such a j minimal. By the induction hypothesis, either $\tau_j > \sigma_j$ or $\tau_j < \sigma_j$ but this implies $\tau > \sigma$ or $\tau < \sigma$.

Case 2 $(\forall j \leq \min(k,l))\tau_j \cong \sigma_j$.

Case 2.1 $k < l$.

Then $\tau < \sigma$.

Case 2.2 $k > l$.

Then $\tau > \sigma$.

Case 2.3 $k = l$.

Then $\tau \cong \sigma$.

□

Theorem 2.2 (Sim)
For all Γ-trees τ, σ, υ,

$$\neg(\tau < \tau)$$

and

$$\tau < \sigma < \upsilon \Rightarrow \tau < \upsilon.$$

Proof
The first assertion follows by induction on $|\tau|$.
First assume that τ is planted.
Suppose that $\tau < \tau$.
Because of the induction hypothesis it is not possible that $\mathrm{Up}(\tau) < \mathrm{Up}(\tau)$ and it is not possible that $\mathrm{Up}(\tau) > \mathrm{Up}(\tau)$, this only leaves the possibility $\mathrm{Up}(\tau) \cong \mathrm{Up}(\tau)$, but then we must have $\mathrm{Down}(\tau) < \mathrm{Down}(\tau)$, which gives a contradiction with the induction hypothesis.

Now suppose that $\tau = \mathrm{Combine}(F)$, with $\{\tau_1, \ldots, \tau_k\}_{\text{multi}}$ a forest of planted Γ-trees. Without loss of generality, assume that $\tau_1 \geq \tau_2 \geq \ldots \geq \tau_k$. Suppose that $\tau < \tau$. This implies the existence of a $1 \leq i \leq k$ such that $\tau_i < \tau_i$, which is in contradiction with the induction hypothesis.

We prove the second assertion by induction on $|\tau| + |\sigma| + |\upsilon|$.

First assume that all three trees are planted. The hypothesis $\tau < \sigma < \upsilon$ produces one of the following seven cases:

Case 1 $\mathrm{Up}(\tau) < \mathrm{Up}(\sigma)$ and $\mathrm{Down}(\tau) < \sigma$, $\mathrm{Up}(\sigma) \leq \mathrm{Up}(\upsilon)$.

By the induction hypothesis we find

$$(\mathrm{Down}(\tau) < \sigma) \wedge (\sigma < \upsilon) \Rightarrow (\mathrm{Down}(\tau) < \upsilon),$$

and then

$$(\mathrm{Up}(\tau) < \mathrm{Up}(\upsilon)) \wedge (\mathrm{Down}(\tau) < \upsilon) \Rightarrow (\tau < \upsilon).$$

Case 2 $\mathrm{Up}(\tau) < \mathrm{Up}(\sigma)$ and $\mathrm{Down}(\tau) < \sigma$, $\mathrm{Up}(\upsilon) < \mathrm{Up}(\sigma)$ and $\sigma \leq \mathrm{Down}(\upsilon)$.

By the previous theorem we find either $\mathrm{Up}(\tau) < \mathrm{Up}(\upsilon)$ or $\mathrm{Up}(\tau) \cong \mathrm{Up}(\upsilon)$ or $\mathrm{Up}(\tau) > \mathrm{Up}(\upsilon)$.

Case 2.1 $\mathrm{Up}(\tau) < \mathrm{Up}(\upsilon)$.

By the induction hypothesis we find

$$(\mathrm{Down}(\tau) < \sigma) \wedge (\sigma < \upsilon) \Rightarrow (\mathrm{Down}(\tau) < \upsilon),$$

and then

$$(\mathrm{Up}(\tau) < \mathrm{Up}(\upsilon)) \wedge (\mathrm{Down}(\tau) < \upsilon) \Rightarrow (\tau < \upsilon).$$

Case 2.2 $\mathrm{Up}(\tau) \cong \mathrm{Up}(\upsilon)$.
By the induction hypothesis we find

$$(\mathrm{Down}(\tau) < \sigma) \wedge (\sigma \leq \mathrm{Down}(\upsilon)) \Rightarrow (\mathrm{Down}(\tau) < \mathrm{Down}(\upsilon)),$$

and then

$$(\mathrm{Up}(\tau) \cong \mathrm{Up}(\upsilon)) \wedge (\mathrm{Down}(\tau) < \mathrm{Down}(\upsilon)) \Rightarrow (\tau < \upsilon).$$

Case 2.3 $\mathrm{Up}(\tau) > \mathrm{Up}(\upsilon)$.
By the induction hypothesis we find

$$(\tau < \sigma) \wedge (\sigma \leq \mathrm{Down}(\upsilon)) \Rightarrow (\tau < \mathrm{Down}(\upsilon))$$

and then

$$(\mathrm{Up}(\upsilon) < \mathrm{Up}(\tau)) \wedge (\tau < \mathrm{Down}(\upsilon)) \Rightarrow (\tau < \upsilon).$$

Case 3 $\mathrm{Up}(\tau) \cong \mathrm{Up}(\sigma)$ and $\mathrm{Down}(\tau) < \mathrm{Down}(\sigma)$, $\mathrm{Up}(\sigma) < \mathrm{Up}(\upsilon)$ and $\mathrm{Down}(\sigma) < \upsilon$.
It follows that $\mathrm{Up}(\tau) < \mathrm{Up}(\upsilon)$.
By the induction hypothesis we find

$$(\mathrm{Down}(\tau) < \mathrm{Down}(\sigma)) \wedge (\mathrm{Down}(\sigma) < \upsilon) \Rightarrow (\mathrm{Down}(\tau) < \upsilon)$$

and then

$$(\mathrm{Up}(\tau) < \mathrm{Up}(\upsilon)) \wedge (\mathrm{Down}(\tau) < \upsilon) \Rightarrow (\tau < \upsilon).$$

Case 4 $\mathrm{Up}(\tau) \cong \mathrm{Up}(\sigma)$ and $\mathrm{Down}(\tau) < \mathrm{Down}(\sigma)$, $\mathrm{Up}(\sigma) \cong \mathrm{Up}(\upsilon)$ and $\mathrm{Down}(\sigma) < \mathrm{Down}(\upsilon)$.
It follows that $\mathrm{Up}(\tau) \cong \mathrm{Up}(\upsilon)$.
By the induction hypothesis we find

$$(\mathrm{Down}(\tau) < \mathrm{Down}(\sigma)) \wedge (\mathrm{Down}(\sigma) < \mathrm{Down}(\upsilon)) \Rightarrow (\mathrm{Down}(\tau) < \mathrm{Down}(\upsilon))$$

and then

$$(\mathrm{Up}(\tau) \cong \mathrm{Up}(\upsilon)) \wedge (\mathrm{Down}(\tau) < \mathrm{Down}(\upsilon)) \Rightarrow (\tau < \upsilon).$$

Case 5 $\mathrm{Up}(\sigma) \leq \mathrm{Up}(\tau)$, $\mathrm{Up}(\upsilon) < \mathrm{Up}(\sigma)$ and $\sigma \leq \mathrm{Down}(\upsilon)$.
By the induction hypothesis we find

$$(\mathrm{Up}(\upsilon) < \mathrm{Up}(\sigma)) \wedge (\mathrm{Up}(\sigma) \leq \mathrm{Up}(\tau)) \Rightarrow (\mathrm{Up}(\upsilon) < \mathrm{Up}(\tau))$$

and

$$(\tau < \sigma) \wedge (\sigma < \mathrm{Down}(\upsilon)) \Rightarrow (\tau < \mathrm{Down}(\upsilon)).$$

Finally,

$$(\mathrm{Up}(\upsilon) < \mathrm{Up}(\tau)) \wedge (\tau < \mathrm{Down}(\upsilon)) \Rightarrow (\tau < \upsilon).$$

Case 6 $\mathrm{Up}(\sigma) < \mathrm{Up}(\tau)$ and $\tau \leq \mathrm{Down}(\sigma)$, $\mathrm{Up}(\sigma) \cong \mathrm{Up}(\upsilon)$ and $\mathrm{Down}(\sigma) < \mathrm{Down}(\upsilon)$.

It follows that $\text{Up}(\upsilon) < \text{Up}(\tau)$.
By the induction hypothesis we find

$$(\tau < \text{Down}(\sigma)) \wedge (\text{Down}(\sigma) < \text{Down}(\upsilon)) \Rightarrow (\tau < \text{Down}(\upsilon))$$

and then

$$(\text{Up}(\upsilon) < \text{Up}(\tau)) \wedge (\tau < \text{Down}(\upsilon)) \Rightarrow (\tau < \upsilon).$$

Case 7 $\text{Up}(\sigma) < \text{Up}(\tau)$ and $\tau \leq \text{Down}(\sigma)$, $\text{Up}(\sigma) < \text{Up}(\upsilon)$ and $\text{Down}(\sigma) < \upsilon$.

By the induction hypothesis we find

$$(\tau \leq \text{Down}(\sigma)) \wedge (\text{Down}(\sigma) < \upsilon) \Rightarrow (\tau < \upsilon).$$

This concludes the proof for planted Γ-trees.

Now suppose that we have trees $\tau = \text{Combine}(F_1)$, $\sigma = \text{Combine}(F_2)$, $\upsilon = \text{Combine}(F_3)$, where $F_1 = \{\tau_1, \ldots, \tau_k\}_{\text{multi}}$, $F_2 = \{\sigma_1, \ldots, \sigma_l\}_{\text{multi}}$ and $F_3 = \{\upsilon_1, \ldots, \upsilon_m\}_{\text{multi}}$ are three forests of planted Γ-trees, let $\tau < \sigma < \upsilon$. Without loss of generality, assume that $\tau_1 \geq \tau_2 \geq \ldots \geq \tau_k$, $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_l$ and $\upsilon_1 \geq \upsilon_2 \geq \ldots \geq \upsilon_m$.
We can consider the following possibilities.

Case 1 $\tau_i \cong \sigma_i \ \forall\ 1 \leq i \leq k$ with $k < l$ and $\sigma_i \cong \upsilon_i \ \forall\ 1 \leq i \leq l$ with $l < m$.

It follows that $\tau_i \cong \upsilon_i \ \forall\ 1 \leq i \leq k$ with $k < m$, so $\tau < \upsilon$.

Case 2 $\tau_i \cong \sigma_i \ \forall\ 1 \leq i \leq k$ with $k < l$ and for certain $j \leq l$, $(\forall i < j)(\sigma_i \cong \upsilon_i)$ and $\sigma_j < \upsilon_j$.

Case 2.1 $j \geq k + 1$.

In this case we have $\tau_i \cong \upsilon_i$ for $1 \leq i \leq k$ with $k < m$ so $\tau < \upsilon$.

Case 2.2 $j \leq k$.

In this case we have $\tau_i \cong \upsilon_i$ for $i < j \leq k$, $\tau_j < \upsilon_j$, so $\tau < \upsilon$.

Case 3 For certain $j \leq k$, $(\forall i < j)(\tau_i \cong \sigma_i)$ and $\tau_j < \sigma_j$ and $\sigma_i \cong \upsilon_i \ \forall\ 1 \leq i \leq l$ with $l < m$.In this case we have $\tau_i \cong \upsilon_i$ for $i < j \leq k$, $\tau_j < \upsilon_j$, so $\tau < \upsilon$.

Case 4 $\tau_i \cong \sigma_i$ for $i < j_1 \leq k$, $\tau_{j_1} < \sigma_{j_1}$ and $\sigma_i \cong \upsilon_i$ for $i < j_2 \leq l$, $\sigma_{j_2} < \upsilon_{j_2}$.
Let $j = \min\{j_1, j_2\}$, then $\tau_i \cong \upsilon_i$ for $i < j \leq k$ and $\tau_j < \upsilon_j$, so $\tau < \upsilon$.

□

Theorem 2.3 (Sim)
For any planted Γ-tree, we have $\text{Down}(\tau) < \tau$ and $\text{Up}(\tau) < \tau$.

Proof
To prove the first assertion we use induction on $|\,\text{Down}(\tau)|$.

Case 1 $\text{Down}(\tau) \cong 0$.

Then the assertion is trivial.

Case 2 $\text{Down}(\tau)$ is a planted Γ-tree.

By the induction hypothesis we have $\text{Down}(\text{Down}(\tau)) < \text{Down}(\tau)$ and $\text{Down}(\text{Down}(\tau)) < \text{Pair}(\text{Up}(\tau), \text{Down}(\text{Down}(\tau)))$.

Using $\text{Down}(\text{Down}(\tau)) < \text{Down}(\tau)$ we obtain
$\text{Pair}(\text{Up}(\tau), \text{Down}(\text{Down}(\tau))) < \tau$.
Hence, $\text{Down}(\text{Down}(\tau)) < \text{Pair}(\text{Up}(\tau), \text{Down}(\text{Down}(\tau))) < \tau$,
and by transitivity (Theorem 2.2), $\text{Down}(\text{Down}(\tau)) < \tau$.

Case 2.1 $\text{Up}(\text{Down}(\tau)) < \text{Up}(\tau)$

$\text{Down}(\text{Down}(\tau)) < \tau$ now implies $\text{Down}(\tau) < \tau$.

Case 2.2 $\text{Up}(\text{Down}(\tau)) \cong \text{Up}(\tau)$

$\text{Down}(\text{Down}(\tau)) < \text{Down}(\tau)$ now implies $\text{Down}(\tau) < \tau$.

Case 2.3 $\text{Up}(\text{Down}(\tau)) > \text{Up}(\tau)$

$\text{Down}(\tau) \leq \text{Down}(\tau)$ now implies $\text{Down}(\tau) < \tau$.

Case 3 $\text{Down}(\tau)$ is the combination of a forest $\{\tau_1, \ldots, \tau_k\}_{\text{multi}}$ of planted Γ-trees. Without loss of generality, assume that $\tau_1 \geq \tau_2 \geq \ldots \geq \tau_k$.

By the induction hypothesis we find $\tau_1 < \text{Pair}(\text{Up}(\tau), \tau_1)$. Since $\tau_1 < \text{Down}(\tau)$ it follows that $\text{Pair}(\text{Up}(\tau), \tau_1) < \tau$, hence $\text{Down}(\tau) < \tau$.

To prove the second assertion we use induction on $|\text{Up}(\tau)|$:

Case 1 $\text{Up}(\tau) \cong 0$.

Then the assertion is trivial.

Case 2 $\text{Up}(\tau) \not\cong 0$ is a planted Γ-tree.

By the induction hypothesis we have $\text{Up}(\text{Up}(\tau)) < \text{Up}(\tau)$ and
$\text{Down}(\text{Up}(\tau)) < \text{Pair}(\text{Down}(\text{Up}(\tau)), \text{Down}(\tau)))$.
By the first part of this theorem we find $(\text{Down}(\text{Up}(\tau)) < \text{Up}(\tau))$ and
$\text{Down}(\tau) < \tau$.
Hence $\text{Pair}(\text{Down}(\text{Up}(\tau)), \text{Down}(\tau)) < \tau$, which gives $\text{Down}(\text{Up}(\tau)) < \tau$.
Now $\text{Up}(\text{Up}(\tau)) < \text{Up}(\tau)$ implies $\text{Up}(\tau) < \tau$.

Case 3 $\text{Up}(\tau) \not\cong 0$ is the combination of a forest of planted Γ-trees. Without loss of generality, assume that $\tau_1 \geq \tau_2 \geq \ldots \geq \tau_k$.

By the induction hypothesis we find $\tau_1 < \text{Pair}(\tau_1, \text{Down}(\tau))$. By the first part of this theorem, $\text{Down}(\tau) < \tau$.
Since $\tau_1 < \text{Up}(\tau)$ it follows that $\text{Pair}(\tau_1, \text{Down}(\tau)) < \tau$. Hence $\tau_1 < \tau$.
It follows that $\text{Up}(\tau) < \tau$.

□

Definition 2.12
The tree λ_k is the Γ-tree with k edges as indicated in Figure 2.5.

Theorem 2.4 (Sim)
For every Γ-tree τ we have $\tau < \lambda_{\text{ht}(\tau)+1}$.

Proof
By induction on $\text{ht}(\tau)$.

Case 1 $\tau \cong 0$.

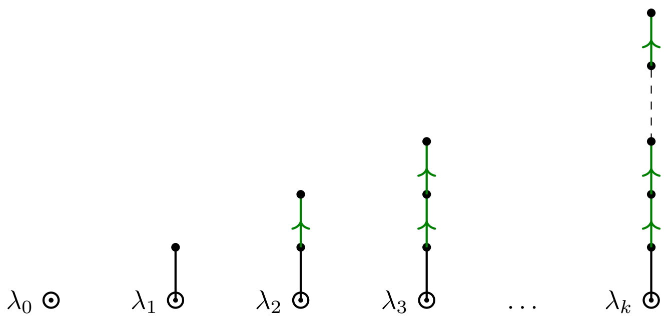
Then the assertion is trivial.

Case 2 $\tau \not\cong 0$ is a planted Γ-tree.

The induction hypothesis learns that

$$\text{Up}(\tau) < \lambda_{\text{ht}(\text{Up}(\tau))+1} \leq \lambda_{\text{ht}(\tau)} = \text{Up}(\lambda_{\text{ht}(\tau)+1})$$

Figure 2.5: The trees λ_k



and

$$\mathrm{Down}(\tau) < \lambda_{\mathrm{ht}(\mathrm{Down}(\tau))+1} < \lambda_{\mathrm{ht}(\tau)+1}.$$

Case 3 τ is the combination of a forest $\tau = \{\tau_1, \ldots, \tau_k\}_{\mathrm{multi}}$ of planted Γ-trees. Without loss of generality, assume that $\tau_1 \geq \tau_2 \geq \ldots \geq \tau_k$.
Since $\tau_1 < \lambda_{\mathrm{ht}(\tau_1)+1} < \lambda_{\mathrm{ht}(\tau)+1}$, we find that $\tau < \lambda_{\mathrm{ht}(\tau)+1}$. □

Lemma 2.1 (Sim)

$$\{\tau \in \Gamma_{\mathrm{tree}} : \tau < \lambda_{k+1}\}$$
$$= \{\mathrm{Combine}(\tau_1, \ldots, \tau_l) : \mathrm{Up}(\tau_i) < \lambda_k, \mathrm{Down}(\tau_i) < \lambda_{k+1} \forall i \in \{1, \ldots, l\}\}.$$

Proof
$\supseteq$ Is clear.
$\subseteq$ Assume τ is a planted Γ-tree for which $\tau < \lambda_{k+1}$.
By definition, we then have $\mathrm{Up}(\tau) < \mathrm{Up}(\lambda_{k+1}) = \mathrm{Up}(\lambda_k)$ and $\mathrm{Down}(\tau) < \lambda_{k+1}$, hence this is the case $l = 1$.
Now assume that τ is the combination of a forest $\tau = \{\tau_1, \ldots, \tau_k\}_{\mathrm{multi}}$ of planted Γ-trees. Without loss of generality, assume that $\tau_1 \geq \tau_2 \geq \ldots \geq \tau_k$. Then $\tau_1 < \lambda_{k+1}$, but then $\tau_i < \lambda_{k+1}$. We find $\mathrm{Up}(\tau_i) < \mathrm{Up}(\lambda_{k+1})$ and $\mathrm{Down}(\tau_i) < \lambda_{k+1}$, hence we find $\mathrm{Up}(\tau_i) < \lambda_k, \mathrm{Down}(\tau_i) < \lambda_{k+1}$ $\forall i \in \{1, \ldots, l\}$.

□

2.3 Counting Γ-trees

Definition 2.13
Let $t(k)$ be the number of Γ-trees with k vertices.

The first terms of the sequence $(t(k))_{k<\omega}$ read as follows

$$0, 1, 1, 3, 10, 39, 160, 702, 3177, 14830, 70678, 342860, 1686486, 8393681, \ldots$$
(OEIS A005750).

In Figure 2.6, the reader can check for her/himself the correctness of the first 5 given terms.

One may note that, starting from $k = 9$, each term is roughly five terms as big as the previous one. In the next chapter, we will show that the asymptotic behaviour of $(t(k))_{k<\omega}$ is indeed of exponential type, with a polynomial term of degree $3/2$ in the denominator[1]

$$t(k) \sim b\frac{q^k}{k^{\frac{3}{2}}}.$$

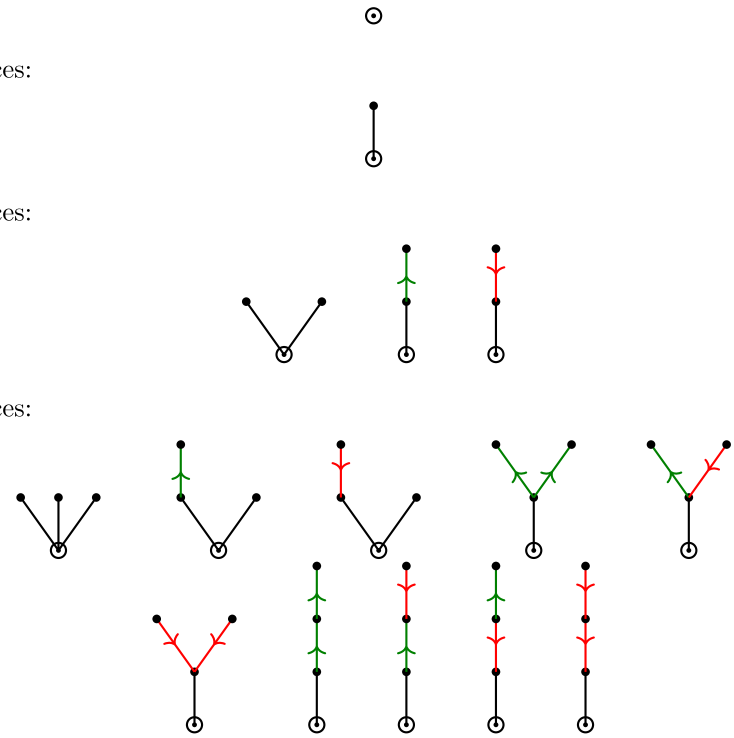
Figure 2.6: The Γ-trees with less than 5 vertices

1 vertex:

2 vertices:

3 vertices:

4 vertices:



Definition 2.14
Let $A(z)$ be the formal power series (Z-transform) corresponding to the sequence $(t(k))_{k<\omega}$, i.e.

$$A(z) = \sum_{k=1}^{+\infty} t(k) z^k.$$

Theorem 2.5 (Sim)

$$A(z) = z \exp\left(\sum_{k=1}^{+\infty} \frac{A(z^k)^2}{kz^k}\right).$$

[1] This behaviour is actually typical for counting rooted trees, see e.g. Bell et al. (2006).

Proof
Let b_n be the number of pairs (τ_1, τ_2) of Γ-trees with $|\tau_1| + |\tau_2| = n + 1$. The generating function of $(b_n)_n$ is then given by $A(z)^2/z$. Next, we observe that $A(z)$ can formally be expanded as an infinite product of formal power series in the following way:

$$A(z) = z \prod_{k=2}^{\infty} \left(\prod_{\substack{(\tau_1,\tau_2)\in\Gamma_{\text{tree}}\times\Gamma_{\text{tree}} \\ \text{with}|\tau_1|+|\tau_2|=k}} (1 - z^{|\tau_1|+|\tau_2|-1})^{-1} \right).$$

Note that the infinite product on the right hand side does indeed define a formal power series because the factors associated to k will only contribute to the coefficients of $z^k, z^{k+1}, \ldots$.
It then follows that the formal power series determined by the right hand side is indeed equal to $A(z)$, since every Γ-tree $\tau = \text{Combine}(\tau_1, \tau_2, \ldots, \tau_l)$ is uniquely defined by the multiset

$$\{(\text{Up}(\tau_1), \text{Down}(\tau_1)), \ldots, (\text{Up}(\tau_l), \text{Down}(\tau_l))\}_{\text{multi}}$$

(remember we are working here up to isomorphism of Γ-trees.)
Grouping terms according to the value of $|\tau_1| + |\tau_2|$, we find

$$\begin{aligned} A(z) &= z \prod_{n=1}^{+\infty} (1 - z^n)^{-b_n} \\ &= z \exp\left(\sum_{n=1}^{+\infty} -b_n \log(1 - z^n) \right). \end{aligned}$$

Expanding the powerseries $\log(1 - z^n)$ and rearranging terms[2], we find

$$\begin{aligned} A(z) &= z \exp\left(\sum_{n=1}^{+\infty} \sum_{k=1}^{+\infty} \frac{b_n z^{nk}}{k} \right) \\ &= z \exp\left(\sum_{k=1}^{+\infty} \sum_{n=1}^{+\infty} \frac{b_n z^{nk}}{k} \right). \end{aligned}$$

Which, by the previous remark on the generating function of $(b_n)_n$, can be written as

$$A(z) = z \exp\left(\sum_{k=1}^{+\infty} \frac{A(z^k)^2}{k z^k} \right).$$

□

By Theorem 2.4, the set $\{\lambda_n : n < \omega\}$ is cofinal in the ordered set of Γ-trees. We will also count, by order, the number of Γ-trees smaller than a fixed λ_n.

Definition 2.15
We define $t_n(k)$ as the number of Γ-trees τ with k vertices and $\tau < \lambda_n$.

[2] This is allowed in the sense of *formal* power series.

Definition 2.16
We define A_n as the formal power series (Z-transform) corresponding to the sequence $(t_n(k))_{k<\omega}$, i.e.

$$A_n(z) = \sum_{k=1}^{+\infty} t_n(k) z^k.$$

Theorem 2.6 (Sim)

$$A_{n+1}(z) = z \exp\left(\sum_{k=1}^{+\infty} \frac{A_n(z^k) A_{n+1}(z^k)}{k z^k}\right).$$

Proof
This proof runs entirely analogously to the previous one. Let b_n^{k+1} be the number of pairs (τ_1, τ_2) of Γ-trees with $\tau_1 < \lambda_{k+1}$, $\tau_2 < \lambda_k$, $|\tau_1| + |\tau_2| = n + 1$. The generating function of $(b_n)_n$ is then given by $A_{k+1}(z) A_k(z)/z$.
Next, we observe that $A(z)$ can formally be expanded as an infinite product of formal power series in the following way:

$$A_{k+1}(z) = z \prod_{k=2}^{+\infty} \left(\prod_{\substack{\tau_1 < \lambda_{k+1}, \tau_2 < \lambda_k \ \Gamma\text{-trees} \\ \text{with } |\tau_1| + |\tau_2| = k}} (1 - z^{|\tau_1| + |\tau_2| - 1})^{-1} \right).$$

Again, we use that every Γ-tree $\tau = \text{Combine}(\tau_1, \tau_2, \dots, \tau_l)$ is uniquely defined by the multiset $\{(\text{Up}(\tau_1), \text{Down}(\tau_1)), \dots, (\text{Up}(\tau_l), \text{Down}(\tau_l))\}_{\text{multi}}$, now in combination with Theorem 2.1.
Grouping terms according to the value of $|\tau_1| + |\tau_2|$, we find

$$\begin{aligned} A_{k+1}(z) &= z \prod_{n=1}^{+\infty} (1 - z^n)^{-b_n^{k+1}} \\ &= z \exp\left(\sum_{n=1}^{+\infty} -b_n^{k+1} \log(1 - z^n)\right) \end{aligned}$$

Expanding the powerseries $\log(1 - z^n)$ and rearranging terms, we find

$$\begin{aligned} A_{k+1}(z) &= z \exp\left(\sum_{n=1}^{+\infty} \sum_{k=1}^{+\infty} \frac{b_n^{k+1} z^{nk}}{k}\right) \\ &= z \exp\left(\sum_{k=1}^{+\infty} \sum_{n=1}^{+\infty} \frac{b_n^{k+1} z^{nk}}{k}\right). \end{aligned}$$

Which, by the previous remark on the generating function of $(b_n^{k+1})_n$, can be written as

$$z \exp\left(\sum_{k=1}^{+\infty} \frac{A_n(z^k) A_{n+1}(z^k)}{k z^k}\right).$$

□

References and Remarks

The results on the ordering of Γ-trees and their proofs are direct translations of the corresponding results that can be found in Schütte (1977) for a similiar ordinal notation system for Γ_0.
The observation that the ordinals below Γ_0 obey elegant functional equations is due to Andreas Weiermann.
Our proofs of Theorems 2.5 and 2.6 follow a standard procedure, see for example Theorem I.1, page 27 of Flajolet and Sedgewick (2009).

A very broad question that might be asked here is if one can alter the definition of Γ-trees and their ordering in a natural way to obtain even larger well-orders. If this turns out to be the case, one could expect to be able to derive corresponding independence results.

3
Asymptotics

This chapter will be somewhat at odds with the others, for it does not (at first sight) discuss mathematical logic. Indeed, in this chapter we will employ techniques from finite combinatorics and complex analysis to proof Theorem 3.11, thus determining the asymptotic behaviour of the sequences $(t(k))_k$ and $(t_n(k))_k$. This will be the main theorem of this chapter. In Chapter 5, we will need this theorem in the derivation of a curious independence result for Sim. It will however be of vital importance for this that the proof of Theorem 3.11 can be formalized in Sim. Needless to say, this will require much more care in formulation and coding. In order not to obscure the arguments (and in convention with common mathematical practice) we will not make this formalisation in Sim precise. The interested reader in the formalisation of analysis in Sim could turn to the canon on reverse mathematics, where Simpson (2009) is a good point of departure.

3.1 Analytic properties of the generating functions $A(z)$ and $A_n(z)$

The expression

$$A(z) = \sum_{k=1}^{+\infty} t(k) z^k$$

determines an analytic function on the open disc $B(0, \rho)$, with ρ the radius of convergence of the series on the right hand side.
Likewise, for $n \geq 1$, the expressions

$$A_n(z) = \sum_{k=1}^{+\infty} t_n(k) z^k$$

determine analytic functions on the open discs $B(0, \rho_n)$, with ρ_n the corresponding radii of convergence.

On the open disc $B(0,\rho)$, the analytic function $A(z)$ satisfies the functional equation derived in Lemma 2.5. To see this, we need to repeat the argument in the proof of Lemma 2.5, checking that each step continues to hold pointwise. Let $z \in B(0,\rho)$ then

$$\begin{aligned}
A(z) &\overset{(*)}{=} \lim_{N\to+\infty} z \prod_{n=1}^{N} (1-z^n)^{-b_n} \\
&= \lim_{N\to+\infty} z \exp\left(-\sum_{n=1}^{N} b_n \log(1-z^n)\right) \\
&= \lim_{N\to+\infty} z \exp\left(\sum_{n=1}^{N}\sum_{k=1}^{+\infty} \frac{b_n z^{nk}}{k}\right) \\
&\overset{(**)}{=} z \exp\left(\sum_{k=1}^{+\infty}\sum_{n=1}^{+\infty} \frac{b_n z^{nk}}{k}\right).
\end{aligned}$$

First note that $\rho \le 1$, since $(t(k))_k$ is clearly unbounded.
The only steps needing further justification are the first and the last one. For $(*)$, we have:

$$|A(z) - z\prod_{n=1}^{N}(1-z^n)^{-b_n}| \le \sum_{k=N+1}^{+\infty} |t(k)z^k| \to 0,$$

by absolute convergence of $A(z)$ in z. Swapping infinite sums in $(**)$ is allowed because

$$\left|\sum_{k=1}^{+\infty}\sum_{n=1}^{+\infty} \frac{b_n z^{nk}}{k}\right| \le \sum_{k=1}^{+\infty} \left|\frac{A(z^k)^2}{kz^k}\right| \le C_z \sum_{k=1}^{+\infty} |z^k| < +\infty,$$

where we made use of the fact that $\dfrac{A(z)}{z}$ is analytic and hence bounded on each closed ball contained in $B(0,\rho)$.
For $k \ge 1$, the analytic function $A_{k+1}(z)$ satisfies on the open disc $B(0,\rho_{k+1})$ the functional equation derived in Theorem 2.6, and the justifications are alike.

We find immediately

$$A_1(z) = z, \tag{3.1}$$

because only the one-vertex tree is smaller than λ_1. Next, we deduce from Theorem 2.1 that a tree is smaller than λ_2 if and only if it does not contain any upwards directed edges. Hence, two Γ-trees smaller than λ_2 are isomorphic if and only if the underlying rooted non-plane trees are isomorphic. The generating function $A_2(z)$ should therefore correspond to the well-known counting function of the rooted non-plane trees studied by Pólya and Otter (see Pólya (1937) and Otter (1948)). Indeed, combining Theorem 2.6 and (3.1), we find back the functional equation

$$A_2(z) = z\exp\left(\sum_{k=1}^{+\infty} \frac{A_2(z^k)}{k}\right),$$

which was deduced and studied by Pólya. The asymptotic analysis in this chapter can therefore be seen as a natural extension of the analysis of Pólya and Otter.

The sequences corresponding to $A_2(z)$, $A_3(z)$ and $A_4(z)$ are respectively

$$0, 1, 1, 2, 4, 9, 20, 48, \ldots \qquad \text{(OEIS A000081)},$$

$$0, 1, 1, 3, 9, 30, 100, 350, \ldots$$

and

$$0, 1, 1, 3, 10, 38, 148, 603, \ldots$$

As a first step in this asymptotic analysis, we obtain crude estimates for the radii of convergence.

When useful, we will also refer to $A(z)$ as $A_\infty(z)$ and to ρ as ρ_∞.

Lemma 3.1
For any $2 \leq n \leq \infty$, the radius of convergence ρ_n of $A_n(z)$ is contained in $]0, 1/2]$.

Proof
From the definition of the numbers $t_n(k)$ and $t(k)$ and from transitivity of the order-relation on the Γ-trees, it follows that for any $2 \leq n \leq l \leq \infty$, the coefficients of $A_l(z)$ dominate those of $A_n(z)$. By consequence, $\rho_l \leq \rho_n$. It therefore suffices to show that $1/2 > \rho_2$ and $\rho > 0$. Let u_k be the number of rooted non-plane trees. The radius of convergence σ of the series

$$U(z) = \sum_{k=1}^{+\infty} u_k z^k$$

is known as Otter's constant[1] and satisfies $\sigma \in [1/4, e^{-1}]$. As we explained above $A_2(z)$ equals $U(z)$, and we have $\sigma = \rho_2$. Furthermore, the trivial estimates $u_k \leq t_k \leq 2^k u_k$ lead to $\rho \in [1/8, e^{-1}]$. □

It should not come as a surprise that the behaviour of $A_n(z)$ at the circle $\partial B(0, \rho_n)$ codes asymptotic information on the sequences $t_n(k)$. In fact, even the radius of this circle is determined by the growth-rate of $t_n(k)$, by the Cauchy root-relation:

$$\rho_n = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|t_n(k)|}}.$$

In order to obtain this behaviour of $A_n(z)$ at $\partial B(0, \rho_n)$, we have to take a closer look at the functional equation satisfied by $A_n(z)$.

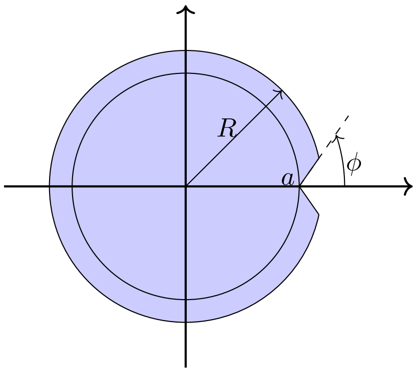
First we define a specific type of region in the complex plain.

Definition 3.1
Let $a \in \mathbb{R}_{>0}$. Given $R \in \,]a, +\infty[$ and $\phi \in \left]0, \frac{\pi}{2}\right[$, the Δ-domain with inlet at a, radius R and angle ϕ is the open domain defined by the intersection of the

[1] For a further discussion on the constant $\sigma = 0.338321856899207\ldots$ see Flajolet and Sedgewick (2009).

open disc $B(0,R)$ and the domain $\mathbb{C}\setminus\{a+re^{i\theta} : \theta\in\ ]-\phi,\phi[\,, r\in[0,+\infty[\}$. See Figure 3.1.
For this domain, we will use the notation $\Delta(\phi,R)$, where a will be made clear by the context.

Figure 3.1: The Δ-domain $\Delta(\phi,R)$ with inlet at a.



We will now prove the following general theorem on analytic functions satisfying a functional equation of the same type as encountered in Lemma 2.6.

Theorem 3.1
Let the power series $S(z)=\sum_{k=1}^{+\infty} s_k z^k$ have radius of convergence $\rho\in]0,1[$ and suppose that $S(z)$ satisfies the following functional equation

$$S(z)=z\exp\left(\sum_{k=1}^{\infty}\frac{g(z^k)S(z^k)}{k}\right)\qquad \forall z\in B(0,\rho), \tag{3.2}$$

where g is analytic on $\overline{B}(0,\rho)$.
Suppose moreover that the following additional conditions are satisfied:

- $s_k\in\mathbb{R}_{>0}\quad \forall k\geq 1$
- all MacLaurin coefficients of g are nonnegative.

Then $S(z)$ is finite on the circle $\partial B(0,\rho)$ and satisfies

$$g(\rho)S(\rho)=1.$$

In addition, $S(z)$ has an analytic continuation to a Δ-region $\Delta(R,\phi)$ with inlet at ρ. In this region, we have

$$S(z)=S(\rho)-c\sqrt{\rho-z}+d(\rho-z)+O((\rho-z)^{3/2})$$

for reals c, d with $c>0$.

We split the proof in several lemmata. It will be useful to write (3.2) as

$$S(z) = z\xi(z)\exp(g(z)S(z)),$$

with

$$\xi(z) = \exp\left(\sum_{k=2}^{+\infty} \frac{g(z^k)S(z^k)}{k}\right).$$

This is convenient, because the part $\xi(z)$ is analytic on the larger disc $B\left(0, \sqrt{\rho}\right)$, as can be seen from the Weierstraß M-test.

Lemma 3.2
$\xi(z)$ is analytic in the open disc $B\left(0, \sqrt{\rho}\right)$.

Proof
Fix $R < \sqrt{\rho}$. Note that the function $\frac{S(z)}{z}$ is analytic on $\overline{B}(0, R^2)$. It follows that there exists a constant $C_R \in \mathbb{R}_{>0}$ such that

$$\left|\frac{g(z^k)S(z^k)}{k}\right| \leq C_R|z^k| \leq C_R R^k$$

holds for every $z \in \overline{B}(0, R)$ and every integer $k \geq 2$.
The result follows using the Weierstraß M-test (note that $R < 1$).

□

Next, we deduce that the series $S(z)$ is finite in the point $z = \rho$.

Lemma 3.3

$$S(\rho) \in \mathbb{R}_{>0}$$

Proof
Suppose $S(\rho)$ diverges, then (because all terms are positive) we would have $S(\rho) = +\infty$ and hence (by Abel's limit theorem) $\lim\limits_{x \underset{<}{\to} \rho} S(x) = +\infty$.

Hence

$$\lim_{x \underset{<}{\to} \rho} (\ln(S(x)) - g(x)S(x)) = \lim_{x \underset{<}{\to} \rho} S(x)\left(\frac{\ln(S(x))}{S(x)} - g(x)\right) = -\infty,$$

implying that

$$\lim_{x \underset{<}{\to} \rho} \frac{S(x)}{\exp(g(x)S(x))} = 0.$$

Using the functional equation (3.2), this yields the contradiction

$$\rho\xi(\rho) = \lim_{x \underset{<}{\to} \rho} x\xi(x) = \lim_{x \underset{<}{\to} \rho} \frac{S(x)}{\exp(g(x)S(x))} = 0.$$

□

We can now see that the equality in (3.2) continues to hold on $\partial B(0, \rho)$. Indeed, we can argue as follows:

- First notice that $S(z)$ is finite on $\partial B(0,\rho)$, because the triangle inequality gives $|S(z)| \leq S(\rho)$ on this circle.
- Any complex number z on this circle can be expressed as a radial limit from inside the disc
$$z = \lim_{\substack{\alpha \to 1 \\ <}} \alpha z.$$
- $\xi(z)$ is continuous in $\overline{B}(0,\rho)$.

Hence, by taking radial limits in (3.2) and applying Abel's limit theorem, we find

$$S(z) = z \exp\left(\sum_{k=1}^{+\infty} \frac{g(z^k)S(z^k)}{k}\right) \qquad \forall z \in \overline{B}(0,\rho). \tag{3.3}$$

3.2 Singularities on the disc of convergence

In this paragraph, we will argue that ρ is a singular point of $S(z)$ in the following sense.

Definition 3.2
For $f : B(a,r) \subseteq \mathbb{C} \to \mathbb{C}$ analytic and $z \in \overline{B}(a,r)$, we say that z is a singular point for f if there does not exist a function h that is analytic on an environment U of z and satisfies $f_{|U \cap B(a,r)} = h_{|U \cap B(a,r)}$.

By the following well-known result, that we state for the sake of completeness, we are certain that the circle $\partial B(0,\rho)$ contains at least one singular point.

Theorem 3.2
If the powerseries $S(z) = \sum_{n=0}^{+\infty} s_n (z-z_0)^n$ has finite radius of convergence ρ, then the circle $\partial B(z_0,\rho)$ contains a singular point of $S(z) : B(z_0,\rho) \to \mathbb{C}$.

Proof
Suppose in desire of contradiction otherwise.
Then, for any $z \in \partial B(z_0,\rho)$, there exists an environment U_z of z and a function g_z analytic on U_z such that $S(z)$ and g_z agree on $B(z_0,\rho) \cap U_z$. By compactness of $\partial B(z_0,\rho)$, there is a finite set of points $z_1,\ldots,z_k$ on $\partial B(z_0,\rho)$ such that $U_{z_1},\ldots,U_{z_k}$ cover $\partial B(z_0,\rho)$. It is easy to see that this implies the existence of an $\varepsilon \in \mathbb{R}_{>0}$ and a $g(z)$ such that

$$B(z_0,\rho+\varepsilon) \subseteq \bigcup_{i \in \{1,\ldots,k\}} U_{z_i}$$

and such that g is analytic on $B(z_0,\rho+\varepsilon)$ (indeed, $g(z) = g_{z_i}(z)$ if $z \in B_{z_i}$ is well-defined). Then the Taylor expansion of g converges on the whole of $B(z_0,\rho+\varepsilon)$ and is necessarily given by $g(z) = \sum_{n=0}^{+\infty} s_n (z-z_0)^n$. Contradiction. □

It follows from the following general theorem by the German mathematician Pringsheim that ρ is a singularity of $S(z)$.

Theorem 3.3 (Pringsheim 1894)
If the sequence $(s_k)_{k<\omega}$ consists of non-negative real numbers and the power-series $S(z) = \sum_{k=0}^{+\infty} s_k z^k$ has finite radius of convergence $\rho \in \mathbb{R}_{>0}$, then ρ is a singular point of $S(z)$.

Proof
Without loss of generality we can suppose that $\rho = 1$. We will use the non-negativity of $(s_k)_{k<\omega}$ under the form

$$|S^{(n)}(z)| \leq S^{(n)}(|z|) \quad \forall n < \omega, \forall z \in B(0, \rho). \tag{3.4}$$

Suppose in desire of contradiction that ρ is not a singular point of $S(z)$.
Then the radius of convergence of the Taylor expansion of $S(z)$ at $z = \frac{1}{2}$ should be strictly larger than $\frac{1}{2}$. Indeed, if it were at most as large as $\frac{1}{2}$, the previous theorem would imply the existence of a singular point in $\overline{B}\left(\frac{1}{2}, \frac{1}{2}\right)$ for $S(z)$. But the only reasonable cadidate, namely $z = 1$, is by our assumption not a singular point of $S(z)$. Hence, $\sum_{n=0}^{+\infty} \frac{S^{(n)}(\frac{1}{2})(z - \frac{1}{2})^n}{n!}$ converges (absolutely) on a ball $B\left(\frac{1}{2}, \frac{1}{2} + \varepsilon\right)$.
By the previous theorem, there is a certain $z_0 \in B(0, 1)$ singular for $S(z)$. However using (3.4) we find that

$$\left|S^{(n)}(\frac{z_0}{2})\right| \leq S^{(n)}\left(\frac{1}{2}\right)$$

and by consequence $\sum_{n=0}^{+\infty} \frac{S^{(n)}(\frac{z_0}{2})(z - \frac{z_0}{2})^n}{n!}$ converges absolutely on $B\left(\frac{z_0}{2}, \frac{1}{2} + \varepsilon\right) \ni z_0$.
This is clearly in contradiction with the singularity of z_0. □

To prove that ρ is in fact the only singularity of $S(z)$ and to determine the specific behaviour of $S(z)$ at ρ, we will use a factorisation result for analytic functions that is known as the Weierstraß Preparation Theorem.

Theorem 3.4 (Weierstraß Preparation Theorem (1895))
Let $F(z, y)$ be a function analytic in both variables at the point (z_0, y_0). Suppose that y_0 is a zero of order k of $y \mapsto F(z_0, y)$, i.e.:

$$\begin{cases} \partial_y^i F(z_0, y_0) = 0 & \forall i < k \\ \partial_y^k F(z_0, y_0) \neq 0. \end{cases}$$

Then there exists a neighbourhood $U \times V$ of (z_0, y_0) and functions $Q_i(z)$ $(i < k)$ and $R(z, y)$ such that

- $R(z, y)$ is analytic and non-zero on $U \times V$,

- $Q_i(z)$ is analytic on U $\forall i < k$,
- $F(z,y) = R(z,y)\left(\sum_{i=0}^{k-1} Q_i(z)y^i + y^k\right)$ on $U \times V$.

Remarks:

1. An analytic function of the form $P(z,y) = \sum_{i=0}^{k-1} Q_i(z)y^i + y^k$ is sometimes called a Weierstraß polynomial.
2. As a surprising consequence, every zero set of the form:
$$\{(z,y) \in \mathbb{C}^2 : F(z,y) = 0\}$$
with $F(z,y)$ analytic at (z_0,y_0), coincide locally around (z_0,y_0) with an *algebraic* zero-set of the form
$$\{(z,y) \in \mathbb{C}^2 : \sum_{i=0}^{k-1} Q_i(z)y^i + y^k = 0\}.$$

Proof
Let $W_z \times W_y$ be a neighbourhood of (z_0,y_0) where $F(z,y)$ is analytic.
Let $B_1 = \overline{B}(y_0,r_1)$ be a closed disc centered at y_0 and contained in W_y with the property that y_0 is the only zero of $F(z_0,y)$ contained in B_1. This is possible since $F(z_0,y)$ is analytic and not constantly zero on W_y.
Let K_z be a compact environment of z_0 contained in W_z.
Then $F(z,y)$ is uniformly continuous on $K_z \times B_1$, so we can choose $r_2 \in \mathbb{R}_{>0}$ such that
$$|F(z_1,y_1) - F(z_2,y_2)| < \min_{y\in\partial B_1} |F(z_0,y)|$$
holds for any $(z_1,y_1),(z_2,y_2) \in K_z \times B_1$ with $\|(z_1,y_1)-(z_2,y_2)\|_2 \leq r_2$.
Let $B_2 = \overline{B}(z_0,r_2)$ after shrinking r_2 if necessary to have B_2 contained in K_z.
For any $z \in B_2$, we define the multiset
$$\text{zero}(z) = \{y \in \overset{\circ}{B}_1 : F(z,y) = 0\}_{\text{multi}}$$
that contains any zero $y \in \overset{\circ}{B}_1$ of $F(z,y)$ of multiplicity l exactly l times. We have for any $z \in B_2$, $\forall y \in \partial B_1$
$$|F(z,y) - F(z_0,y)| < F(z_0,y) \tag{3.5}$$
and then by Rouché 's theorem:
$$|\text{zero}(z)| = |\text{zero}(z_0)| = k.$$
We can now write down our candidate for the Weierstraß polynomial in the factorisation of $F(z,y)$.
Define
$$P(z,y) = \prod_{y_i \in \text{zero}(z)} (y - y_i)$$

and

$$S_m(z) = \sum_{y_i \in \text{zero}(z)} y_i^m \qquad m \in \{1, \dots, k\}.$$

Then $P(z, y)$ is a polynomial in y with the coefficient of each power y^i depending on z. Let $Q_i(z)$ be the function giving the coefficient of y^i in $P(z, y)$ as a function of z.

We have to prove that all $Q_i(z)$ are analytic in neighbourhoods of z_0. However, it is a well-known algebraic property that the functions $Q_i(z)$ (which are exactly the elementary symmetric polynomials evaluated in zero(z)) can be written as polynomials in the functions $S_m(z)$. Hence, it suffices to investigate analyticity of $S_m(z)$ around z_0. For this, we rewrite $S_m(z)$, on the ball B_2, as the following integral:

$$S_m(z) = \frac{1}{2\pi i} \int\limits_{\partial B_1} y^m \frac{\partial_y F(z, y)}{F(z, y)} dy, \qquad \forall z \in B_2. \tag{3.6}$$

This equality follows from the Residue Theorem, after making the following remarks:

- By (3.5), $F(z, y)$ does not have any zeroes on $B_2 \times \partial B_1$, so the integrand is as a function of y analytic on B_1, with exception of certain poles in $\overset{\circ}{B}_1$ induced by zeroes of $F(z, y)$.
- Every zero y_i of multiplicity l of $F(z, y)$ induces the residue $l y_i^m$ of $y^m \dfrac{\partial_y F(z, y)}{F(z, y)}$ at the pole $y = y_i$.

We go on to observe that the integral in (3.6) can be differentiated with respect to z under the integral sign in $\overset{\circ}{B}_2$ (the integrand is continuous on $\overset{\circ}{B}_2 \times \partial B_1$ and for each fixed $y \in \partial B_1$ analytic as a function of z on $\overset{\circ}{B}_2$).

By consequence $S_m(z)$ and $Q_m(z)$ are analytic on $\overset{\circ}{B}_2$.

The choice for $P(z, y)$ in the role of the Weierstraß polynomial in the factorisation of $F(z, y)$ therefore appears to be a right one, the more since it is clear now that

$$R(z, y) := \frac{F(z, y)}{P(z, y)}$$

has no zeroes in $B_2 \times B_1$.

For fixed $z \in B_2$, $R(z, y)$ is an analytic function of y in $\overset{\circ}{B}_1$, because it consists of an analytic function divided by a polynomial and the zeroes of the denominator are all matched by zeroes in the nominator.

Now we prove that for fixed $y \in \overset{\circ}{B}_1$, $R(z, y)$ is an analytic function of z in $\overset{\circ}{B}_2$.

Fix $y \in \overset{\circ}{B}_1$.

We consider the auxiliary function $\Phi_y(z, u) = \dfrac{F(z, u)}{P(z, u)(u - y)}$ on $B_2 \times B_1$.

We first observe that the following claim holds.

For any closed subset K of $\mathring{B}_2$, there exists a circle $\partial B(y_0, r_K) \subseteq \mathring{B}_1$ such that $y \in B(y_0, r_K)$ and $P(z, u)$ is non-zero on $K \times \partial B(y_0, r_K)$.

To prove this observation, we recall that $P(z, u)$ is non-zero on $K \times \partial B_1$ for any closed subset K of $\mathring{B}_2$, as well as continuous on $K \times B_1$ (the $Q_i(z)$ are even analytic on $\mathring{B}_2$).
By uniform continuity on $K \times B_1$, we can choose $r_K < r_1$ so large that both

$$|y - y_0| < r_K$$

and for each $(z, u) \in K \times \partial B(y_0, r_K)$, there is $u' \in \partial B_1$ such that

$$|P(z, u) - P(z, u')| < \min_{\substack{u \in \partial B_1 \\ z \in K}} |P(z, u)|.$$

This choice of circle $B(y_0, r_K)$ ensures $|P(z, u)| \neq 0$.
When K and $\partial B(y_0, r_K)$ are chosen in this way, $\Phi_y(z, u)$ is continuous on $K \times \partial B(y_0, r_K)$.

Moreover, $\Phi_y(z, u)$ is even analytic as a function of z on $\mathring{K}$ for fixed $u \in \partial B(y_0, r_K)$, since the denominator will in this case never vanish.
By differentiation under the integral-sign, we can thus conclude that the function

$$G(z) = \frac{1}{2\pi i} \int\limits_{\partial B(y_0, r_K)} \Phi_y(z, u) du$$

is analytic in z on $\mathring{K}$.
Upon closer inspection of this integral, we find that its integrand

$$\Phi_y(z, u) = \frac{F(z, u)}{P(z, u)(u - y)}$$

is in fact meromorphic (as a function of u) on $\overline{B}(y_0, r_K)$ with a sole pole in $u = y$. Hence, the integral $G(z)$ evaluates to $R(z, y)$ on $\mathring{K}$ (again using the Residue Theorem) and we proved that $R(z, y)$ is analytic on the interior of every compactum contained in $\mathring{B}_2$ and by consequence it is analytic in $\mathring{B}_2$.

We can now also check that $R(z, y)$ is continuous in (z, y), for $z \in \mathring{K}$. Indeed, for all y' in a compact, sufficiently small neighbourhood C of y and for all z_1, $z_2 \in \mathring{K}$, we find

$$|G(z_1, y) - G(z_2, y')| = \left| \frac{1}{2\pi i} \int\limits_{\partial B(y_0, r_K)} \Phi_y(z_1, u) du - \frac{1}{2\pi i} \int\limits_{\partial B(y_0, r_K)} \Phi_{y'}(z_2, u) du \right|$$

$$\leq r_K \cdot \max_{u \in \partial B(y_0, r_K)} |\Phi_y(z_1, u) - \Phi_{y'}(z_2, u)|.$$

Since $\Phi_y(z, u)$ is continuous as a function of the three variables (y, z, u), it is uniformly continuous on the compact set $C \times K \times \partial B(y_0, r_K)$ and this suffices to completes the proof. □

The Implicit Function Theorem for analytic functions is a direct consequence of the Weierstraß preparation theorem.

Theorem 3.5 (Analytic Implicit Function Theorem (AIFT))
Let $F(z, y)$ be analytic in both variables at the point (z_0, y_0).
If $F(z_0, y_0) = 0 \neq \partial_y F(z_0, y_0)$, then there exist a neighbourhood U_{z_0} of z_0 and a unique analytic function $y(z)$ on U_{z_0} such that $F(z, y(z)) = 0\ \forall z \in U$ and $y(z_0) = y_0$.
In addition, there exists a neighbourhood V_{y_0} of y_0 such that

$$(\forall (z, y) \in U_{z_0} \times V_{y_0})(F(z, y) \Rightarrow y = y(z)).$$

Proof
Applying the Weierstraß Preparation Theorem for $k = 1$, we find a neighbourhood $U_{z_0} \times V_{y_0}$ of (z_0, y_0) and functions $P(z, y)$, $R(z, y)$, analytic on $U_{z_0} \times V_{y_0}$ such that

- $F(z, y) = P(z, y)R(z, y)$ on $U_{z_0} \times V_{y_0}$,
- $R(z, y) \neq 0$ on $U_{z_0} \times V_{y_0}$,
- $P(z, y) = Q_0(z) + y$, with $Q_0(z)$ analytic on U_{z_0}.

It is then clear that the function $y(z) = -Q_0(z)$ and the neighbourhoods U_{z_0}, V_{y_0} satisfy the requirements in the assertion of this lemma. □

Define
$$H(z, y) = z\xi(z) \exp(g(z)y).$$
We will gain valuable information by applying the AIFT (or rather its contraposition) on the analytic function (in $\overline{B}(0, \rho)$)

$$G(z, y) = H(z, y) - y.$$

Lemma 3.4
$$g(\rho)S(\rho) = 1 \tag{3.7}$$

Proof
If $\partial_y G(\rho, S(\rho))$ were non-zero, we could apply the AIFT to find a holomorphic extension of $S(z)$ to a neighbourhood of ρ. This would contradict Pringsheim's Theorem which asserts that ρ is a singular point of $S(z)$.
Hence $0 = \partial_y G(\rho, S(\rho)) = g(\rho)\underbrace{\xi(\rho) \exp(g(\rho)S(\rho))}_{S(\rho)} - 1 = g(\rho)S(\rho) - 1.$ □

Lemma 3.5

1. $\partial_y G(z, S(z)) \neq 0$ on $z \in \partial\overline{B}(0, \rho) \setminus \{\rho\}$
2. $\partial_{yy} G(\rho, S(\rho)) > 0$

Proof
For the first point we remark that since $g(z)$ and $S(z)$ have non-negative, respectively strictly positive (for $n \geq 1$), Maclaurin coefficients, we have

$$\left.\begin{array}{l}|g(z)| \leq g(\rho)\\ |S(z)| < S(\rho)\end{array}\right\} \Rightarrow |g(z)S(z)| < 1.$$

The second point follows from the following easy calculation.

$$\partial_{yy}G(z,y) = (g(z))^2 z\xi(z)\exp(g(z)y) \Rightarrow \partial_{yy}G(\rho, S(\rho)) > 0.$$

□

Corollary 3.1
ρ is the only singularity of $S(z)$ on the circle $\partial B(0,\rho)$.

Proof
Let $z_0 \in \partial B(0,\rho) \setminus \{\rho\}$. By the first point of the previous lemma, the AIFT is applicable to the function $G(z,y)$ in the point $(z_0, S(z_0))$. Hence, there exist U_{z_0}, V_{z_0} and $f(z)$ analytic in U_{z_0} such that $\{(z, f(z)) : z \in U_{z_0}\}$ provides the only solutions in $U_{z_0} \times V_{z_0}$ for $G(z,y) = 0$. As $S(z)$ is continuous on the radius from 0 to z_0, we have $\{(\alpha z_0, f(\alpha z_0)) : \alpha \in [1-\varepsilon, 1[\} \subseteq U_{z_0} \times V_{z_0}$ for ε small enough. By consequence $f(z)$ and $S(z)$ will agree on $B(0,\rho) \cap U_{z_0}$ and z_0 is not singular. □

It then follows from a compactness argument just like in the proof of Theorem 3.2, that $S(z)$ has an analytic extension to a Δ-region $\Delta(R,\phi)$.

We can now complete the proof of Theorem 3.1, by proving Lemma 3.6, the conditions of which are satisfied because of Lemma 3.4 and Lemma 3.5.

Lemma 3.6
Let $S(z)$ be analytic on a Δ-region $\Delta(R,\phi)$, with inlet at ρ, having non-negative MacLaurin coefficients and satisfying $F(z, S(z)) = 0$ on $\overline{B}(0,\rho)$, where $F(z,y)$ is a function analytic in both variables at the point $(\rho, S(\rho))$.
Suppose that

- $F(\rho, S(\rho)) = \partial_y F(\rho, S(\rho)) = 0$,
- $\partial_{yy}F(\rho, y_0) > 0$ and
- $\partial_z F(\rho, S(\rho)) > 0$.

Then there is $c \in \mathbb{R}_{>0}$, $d \in \mathbb{R}$ such that

$$S(z) = S(\rho) - c\sqrt{\rho - z} + d(\rho - z) + O((\rho - z)^{3/2}),$$

holds on $\Delta(R,\phi)$.

In fact, c is given explicitly by $c = \sqrt{\dfrac{2\,\partial_z F(\rho, S(\rho))}{\partial_{yy}F(\rho, S(\rho))}}$.

Proof
Choose $Q_0(z)$, $Q_1(z)$, $R(z,y)$ holomorphic in a neighbourhood $U \times V$ of $(\rho, S(\rho))$ such that $R(z,y)$ has no zeroes in this region and $F(z,y)$ admits the factorisation

$$F(z,y) = (Q_0(z) + Q_1(z)y + y^2).R(z,y).$$

Then

$$Q_0(z) + Q_1(z)S(z) + S(z)^2 = 0 \tag{3.8}$$

holds in the intersection of $\overline{B}(0,\rho)$ with the neighbourhood U of ρ.
Let $D(z) = Q_1(z)^2 - 4Q_0(z)$ be the discriminant of (3.8). By the usual algebraic manipulations, we rewrite (3.8) to

$$\left(S(z) + \frac{1}{2}Q_1(z)\right)^2 = \frac{1}{4}D(z)$$

on $B(0,\rho) \cap U$.
Since

$$\partial_y F(z,y) = (Q_1(z) + 2y).R(z,y) + (Q_0(z) + Q_1(z)y + y^2)\partial_y R(z,y)$$

and $\partial_y F(\rho, S(\rho)) = 0$, we obtain

$$(Q_1(\rho) + 2S(\rho)).R(\rho, S(\rho) + (Q_0(\rho) + Q_1(\rho)S(\rho) + S(\rho)^2)\partial_y R(\rho, S(\rho)) = 0.$$

Since $Q_0(\rho) + Q_1(\rho)S(\rho) + S(\rho)^2 = 0$ and $R(\rho, S(\rho)) \neq 0$, we find

$$Q_1(\rho) + 2S(\rho) = 0.$$

It then follows that

$$D(\rho) = 0.$$

Since

$$\partial_{yy}F(z,y) = 2.R(z,y) + 2(Q_1(z)+2y).\partial_y R(z,y) + (Q_0(z)+Q_1(z)y+y^2)\partial_{yy}R(z,y),$$

we obtain

$$\partial_{yy}F(\rho, S(\rho)) = 2R(\rho, S(\rho)).$$

Further,

$$\partial_z F(z,y) = (Q_0'(z) + Q_1'(z).y)R(z,y) + (Q_0(z) + Q_1(z)y + y^2)\partial_z R(z,y)$$

leads to

$$\partial_z F(\rho, S(\rho)) = (Q_0'(\rho) + Q_1'(\rho).S(\rho))R(\rho, S(\rho)) + (Q_0(\rho) + Q_1(\rho)S(\rho + S(\rho)^2)\partial_z R(\rho, S(\rho)).$$

Since the second term equals zero, we get

$$\partial_z F(\rho, S(\rho)) = (Q_0'(\rho) + Q_1'(\rho).S(\rho))R(\rho, S(\rho))$$

and since $Q_1(\rho) + 2S(\rho) = 0$ this becomes

$$\begin{aligned}\partial_z F(\rho, S(\rho)) &= (Q_0'(\rho) + Q_1'(\rho).\frac{-Q_1(\rho)}{2})R(\rho, S(\rho)) \\ &= \left(Q_0'(\rho) - \frac{1}{2}Q_1(\rho)Q_1'(\rho)\right)\frac{1}{2}\partial_{yy}F(\rho, S(\rho)),\end{aligned}$$

so

$$\frac{1}{2}Q_0'(\rho) - \frac{1}{4}Q_1(\rho)Q_1'(\rho)) = \frac{\partial_z F(\rho, S(\rho))}{\partial_{yy} F(\rho, S(\rho))}.$$

From

$$D(z) = Q_1(z)^2 - 4Q_0(z)$$

we find

$$D'(z) = 2Q_1(z)Q_1'(z) - 4Q_0'(z)$$

and thus

$$D'(\rho) = -\frac{8\,\partial_z F(\rho, S(\rho))}{\partial_{yy} F(\rho, S(\rho))} < 0.$$

Hence we can write

$$D(z) = (\rho - z)h_1(z)$$

with $h_1(z) \neq 0$ on an environment of ρ.
By shrinking U if necessary we find a square root $h(z)$ of $h_1(z)$ on U.
It follows that both $S(z) + \frac{1}{2}Q_1(z)$ and $\frac{h(z)}{2}\sqrt{\rho - z}$ are square roots of $\frac{D(z)}{4}$ on $B(0, \rho) \cap U$.
Hence, by changing $h(z)$ to $-h(z)$ if necessary, we find on $B(0, \rho) \cap U$

$$S(z) = \frac{-Q_1(z) + h(z)\sqrt{\rho - z}}{2}.$$

It follows that

$$S(z) = S(\rho) + \frac{h(\rho)}{2}\sqrt{\rho - z} - \frac{Q_1'(\rho)}{2}(\rho - z) + O((\rho - z)^{3/2})$$

with

$$\frac{h(\rho)}{2} = \pm\sqrt{\frac{2\,\partial_z F(\rho, S(\rho))}{\partial_{yy} F(\rho, S(\rho))}} \neq 0.$$

It is clear that in fact $h(\rho) < 0$ (and hence the minus sign should be chosen), for otherwise we would find

$$h(\rho)\sqrt{\rho - z} \leq |S(z) - S(\rho) - \frac{h(\rho)}{2}\sqrt{\rho - z}| = O(\rho - z),$$

since $S(z)$ is increasing on $[0, \rho[$, but this is clearly contradictory. □

This completes the proof of Theorem 3.1.

Using Theorem 3.1, we gain the following information on the functions $A_n(z)$.

Theorem 3.6
$\forall n \geq 2$

(1) $A_n(z)$ has an analytic continuation to a Δ-region with inlet at ρ_n, where it satisfies

$$A_n(z) = A_n(\rho_n) - c_n\sqrt{\rho_n - z} + d_n(\rho_n - z) + O((\rho_n - z)^{3/2})$$

(2) $A_{n-1}(\rho_n)A_n(\rho_n) = \rho_n$,

(3) $\rho_{n-1} < \rho_n$.

Proof
By induction on n.
For $n = 2$, (3) is trivially true since $\rho_1 = +\infty$ and $\rho_2 < 1$.
(1) and (2) follow immediately by Theorem 3.1, with $g(z) = 1$.
Next, we prove the theorem for $n+1$, assuming the validity for n.
Recall that $\rho_n \leq \rho_{n+1}$. It suffices to exclude $\rho_{n+1} = \rho_n$. Indeed, once we have proved (3), the other points follow by applying Theorem 3.1 with $g(n) = \dfrac{A_n(z)}{z}$ (which is then certainly analytic at $\overline{B}(0, \rho_{n+1})$).
Suppose in desire of contradiction that $\rho_{n+1} = \rho_n$.
We again consider $H(z, y) = z\xi(z)\exp(g(z)y)$ with $g(z) = \dfrac{A_n(z)}{z}$ and

$$\xi(z) = \exp\left(\sum_{k=2}^{+\infty} \frac{g(z^k)A_{n+1}(z^k)}{k}\right).$$

Repeating the first steps in the proof of Theorem 3.1, we find that ξ is analytic in $B(0, \sqrt{\rho_n})$, and that $H(z, A_{n+1}(z)) = A_{n+1}(z)$ holds on $\overline{B}(0, \rho_n)$.
Next we consider the real function

$$h : [0, \rho_n] \to \mathbb{R}_{>0} : x \mapsto x\xi(x)\exp(g(x)A_{n+1}(x))g(x).$$

We claim that $h(\rho_n) > 1$. Indeed,

$$h(\rho_n) = H(\rho_n, A_{n+1}(\rho_n))g(\rho_n) = A_{n+1}(\rho_n)\frac{A_n(\rho_n)}{\rho_n} > A_n(\rho_n)\frac{A_{n-1}(\rho_n)}{\rho_n} = 1.$$

Since h is strictly increasing and continuous, there is $0 < r < \rho_n$ such that $h(r) = 1$. Then $H(z, y)$ is analytic in both variables in an environment of $(r, A_{n+1}(r))$ and a calculation immediately yields

$$\partial_y H(r, A_{n+1}(r)) = h(r) = 1.$$

Using the chain rule, we obtain

$$\begin{aligned} A'_{n+1}(r) &= \partial_z H(r, A_{n+1}(r)) + \partial_y H(r, A_{n+1}(r))A'_{n+1}(r) \\ &= \partial_z H(r, A_{n+1}(r)) + A'_{n+1}(r) \end{aligned}$$

Hence $\partial_z H(r, A_{n+1}(r)) = 0$.
However, direct calculation of $\partial_z H(r, A_{n+1}(r))$ gives a sum of strictly positive terms, so we have arrived at a contradiction. □

Although $A(z)$ will exhibit the same square-root-type singular behaviour at ρ, this does not directly follow from Theorem 3.1. $A(z)$ does in fact satisfy equation (3.2) in the statement of this theorem with $g(z) = \dfrac{A(z)}{z}$.
The function $g(z) = \dfrac{A(z)}{z}$ is analytic on $B(0, \rho)$, but there is no way to extend $g(z)$ to an analytic function on $\overline{B}(0, \rho)$ (by Theorem 3.2).
As we will see, the argument from Theorem 3.1 will nevertheless need only minor modifications.

Theorem 3.7
$A(z)$ is finite on the circle $\partial B(0,\rho)$ and satisfies

$$A(\rho) = \sqrt{\frac{\rho}{2}}. \tag{3.9}$$

In addition, $A(z)$ has an analytic continuation to a Δ-region with inlet at ρ. In this region, we have

$$A(z) = A(\rho) - c\sqrt{\rho - z} + d(\rho - z) + O((\rho - z)^{3/2}),$$

with $c \in \mathbb{R}_{>0}$ and $d \in \mathbb{R}$.

Proof
$A(z)$ satisfies

$$A(z) = z \exp\left(\sum_{k=1}^{+\infty} \frac{g(z^k)A(z^k)}{k}\right), \qquad \forall z \in B(0,\rho),$$

with $g(z) = \dfrac{A(z)}{z}$. Upon examination of the proof of Theorem 11, we find that we did not need analyticity of $g(z)$ on $\overline{B}(0,z)$ until the proof of Lemma 23. Indeed, in this first part of the proof, we only use analyticity of $g(z)$ on $B(0,z)$, together with radial continuity of g on $\overline{B}(0,z)$ (which follows for $\dfrac{A(z)}{z}$ by Abel's limit theorem).
Hence, we find that $A(\rho) \in \mathbb{R}_{>0}$ and that

$$A(z) = z \exp\left(\sum_{k=1}^{+\infty} \frac{g(z^k)A(z^k)}{k}\right), \qquad \forall z \in \overline{B}(0,\rho).$$

By Pringsheim's Theorem, ρ is a singular point of $A(z)$.
To continue we need a functional equation that is analytic in z on $\overline{B}(0,\rho)$. This problem is easily solved by considering the shifted powerseries

$$B(z) = \sum_{k=1}^{+\infty} t(k)z^{k-1} = \frac{A(z)}{z}.$$

It follows that

$$B(z) = \exp\left(\sum_{k=1}^{+\infty} \frac{B(z^k)^2 z^k}{k}\right)$$

on $\overline{B}(0,\rho)$.
We consider

$$H(z,y) = \xi(z)\exp(y^2 z),$$

with

$$\xi(z) = \exp\left(\sum_{k=2}^{+\infty} \frac{B(z^k)^2 z^k}{k}\right) = \exp\left(\sum_{k=2}^{+\infty} \frac{A(z^k)^2}{z^k k}\right).$$

Then $\xi(z)$ is analytic on $B(0,\sqrt{\rho})$ and $H(z,y)$ is analytic in both variables on $B(0,\sqrt{\rho}) \times \mathbb{C}$. Moreover, putting again

$$G(z,y) = H(z,y) - y,$$

we have

$$G(z, B(z)) = 0$$

on $\overline{B}(0, \rho)$.
Just as in Lemma 3.4 we find by the fact that ρ is a singular point, in combination with the AIFT that

$$\partial_y G(\rho, B(\rho)) = 0.$$

The partial derivative to y is given by

$$\partial_y G(z, y) = -1 + 2yz(y + G(z, y)).$$

Hence we find from $\partial_y G(\rho, B(\rho)) = 0$:

$$-1 + 2B(\rho)\rho(B(\rho) + G(\rho, B(\rho)) = 0$$

Since $G(\rho, B(\rho)) = 0$, this leads to

$$B(\rho) = \sqrt{\frac{1}{2\rho}},$$

and

$$A(\rho) = \sqrt{\frac{\rho}{2}}.$$

Since

$$|zB^2(z)| < \rho B^2(\rho) = \frac{1}{2},$$

we find by the same argument as explained in Corollary 3.1, that there are no other singular points for $B(z)$ on $\partial B(0, \rho)$ than ρ. Hence, $A(z)$ can be analytically continued to a Δ-region with inlet at ρ.
It is easy verifying that

$$\partial_{yy} G(\rho, A(\rho)) > 0 \quad \text{and} \quad \partial_z G(\rho, A(\rho)) > 0.$$

We can apply Lemma 3.6 to finish the proof. □

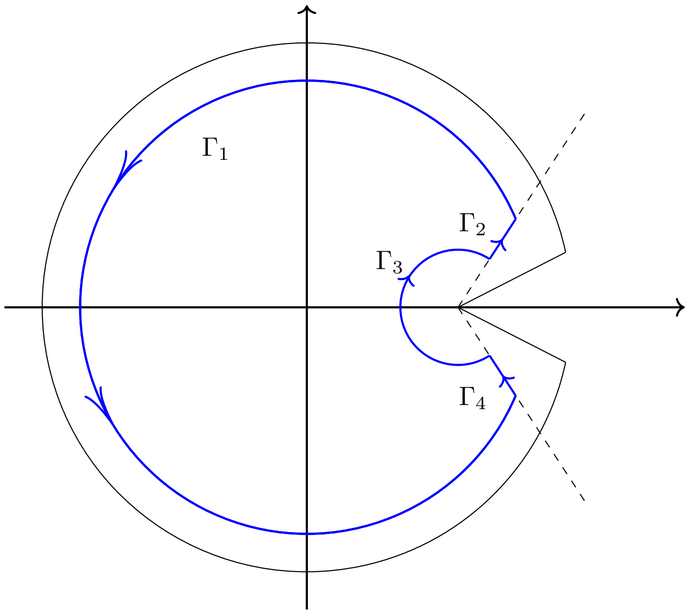
3.3 Asymptotics of the sequences $(t(k))_k$ and $(t_n(k))_k$

Asymptotic information on a sequence $(s_k)_k$ can be obtained from the singular expansion of the generating function $S(z)$ by integrating $S(z)$ on a suitably chosen contour. We illustrate this argument in the following theorem.

Theorem 3.8 (Transfer Theorem)
Let f be a function that is analytic in a Δ-domain $\Delta(R, \phi)$ with inlet at ρ with radius $R \in \,]\rho, +\infty[$ and angle $\phi \in \left]0, \frac{\pi}{2}\right[$.
Let $\alpha \in \mathbb{R}$ arbitrary.
If $f(z) = O((\rho - z)^\alpha)$ holds on $\Delta(R, \phi)$, then

$$f^{(n)}(0) = O\left(\frac{\rho^{-n} n!}{n^{\alpha+1}}\right).$$

Figure 3.2: Contour line



Proof

We first consider the special case $\rho = 1$.

Choose $r \in \,]1, R[$ and $\phi < \psi < \dfrac{\pi}{2}$ arbitrarily.

For $n < \omega$, we consider the contour $\Gamma^{(n)} = \Gamma_1^{(n)} \cup \Gamma_2^{(n)} \cup \Gamma_3^{(n)} \cup \Gamma_4^{(n)}$, defined by

$$\begin{aligned}
\Gamma_1^{(n)} &= \left\{ 1 + \frac{e^{i\theta}}{n} : \theta \in [\psi, 2\pi - \psi] \right\} \\
\Gamma_2^{(n)} &= \left\{ 1 + te^{i\psi} : t \in \left[\frac{1}{n}, -\cos\psi + \sqrt{r^2 - \sin^2\psi} \right] \right\} \\
\Gamma_3^{(n)} &= B(0, r) \setminus \left\{ 1 + re^{i\theta} : \theta \in [-\psi, \psi], r \in [0, +\infty[\right\} \\
\Gamma_4^{(n)} &= \left\{ 1 + te^{-i\psi} : t \in \left[\frac{1}{n}, -\cos\psi + \sqrt{r^2 - \sin^2\psi} \right] \right\},
\end{aligned}$$

where orientations are as in Figure 3.2.

By Cauchy's formula,

$$\frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \int_{\Gamma^{(n)}} \frac{f(z)dz}{z^{n+1}},$$

where the contour is traversed anti-clockwise.

We bound every term

$$I_i = \frac{1}{2\pi i} \int_{\Gamma_i^{(n)}} \frac{f(z)dz}{z^{n+1}}$$

by an $O(n^{-\alpha-1})$-term.
This is a simple matter for the circle-arcs $\Gamma_1^{(n)}$ and $\Gamma_3^{(n)}$. Indeed,

$$\begin{aligned}
|I_3| &\leq \frac{1}{2\pi} \,.\, \max_{z\in\Gamma_1^{(n)}} \left|\frac{f(z)}{z^{n+1}}\right| \,.\, \frac{2\pi}{n} \\
&\leq \max_{z\in\overline{B}(1,\frac{1}{n})} |f(z)| \,.\, \max_{z\in\overline{B}(1,\frac{1}{n})} |z^{-n-1}| \,.\, \frac{2\pi}{n} \\
&\leq O(n^{-\alpha}) \,.\, O(1) \,.\, \frac{2\pi}{n} \\
&= O(n^{-\alpha-1}),
\end{aligned}$$

for n large enough.
And

$$\begin{aligned}
|I_1| &\leq \frac{1}{2\pi} \,.\, \max_{z\in\Gamma_3^{(n)}} |f(z)| \,.\, \max_{z\in\Gamma_3^{(n)}} \left|\frac{1}{z^{n+1}}\right| \,.\, 2\pi r \\
&\leq \frac{1}{2\pi} \,.\, O(1) \,.\, O\left(\frac{1}{r^{n+1}}\right) \,.\, 2\pi r \\
&= O(r^{-n}).
\end{aligned}$$

Next, we bound the integral over $\Gamma_2^{(n)}$,

$$\begin{aligned}
|I_2| &\leq \frac{1}{2\pi} \int_{\frac{1}{n}}^{r'} \frac{|f(1+te^{i\psi})|}{|1+te^{i\psi}|^{n+1}} dt \\
&\leq \frac{C}{2\pi} \int_{\frac{1}{n}}^{r'} \frac{t^\alpha}{|1+te^{i\psi}|^{n+1}} dt \\
&= \frac{C}{2\pi} \int_{1}^{nr'} \frac{n^{-\alpha-1}s^\alpha}{|1+\frac{s}{n}e^{i\psi}|^{n+1}} ds \\
&\leq O(n^{-\alpha-1}) \,.\, \int_1^{+\infty} \frac{s^\alpha}{|1+\frac{se^{i\psi}}{n}|^{n+1}} ds \\
&\leq O(n^{-\alpha-1}) \,.\, \int_1^{+\infty} \frac{s^\alpha}{(1+\frac{s\cos\psi}{n})^{n+1}} ds.
\end{aligned}$$

Hence, it suffices to bound, for all $n \geq n_0$ the function

$$g_n(s) = \frac{s^\alpha}{(1+\frac{s\cos\psi}{n})^{n+1}}$$

by a function that is integrable on $[1, +\infty[$ and independent from n.
Note however that (since $0 < \psi < \frac{\pi}{2}$)

$$g_n(s) = \frac{s^\alpha}{(1+\frac{s\cos\psi}{n})^{n+1}} < \frac{s^\alpha}{(1+\frac{s\cos\psi}{n})^{n}} =: h_n(s).$$

Since $h_n(s)$ is, for fixed $s \in [1, \infty[$, decreasing[2] as a function of $n < \omega$ and moreover h_N is integrable for N large enough, we can simply choose

$$g(s) = h_N(s).$$

[2]This becomes apparent by calculating the second derivative with respect to y of the function $(1 + \frac{a}{y})^{-y}$, where $a \in \mathbb{R}_{>0}$.

The argument for showing that $|I_4| \leq O(n^{-\alpha-1})$ is related. Hence

$$\frac{f^{(n)}(0)}{n!} = +O(r^{-n}) + O(n^{-\alpha-1}) + O(n^{-\alpha-1}) + O(n^{-\alpha-1}) = O(n^{-\alpha-1}).$$

This proves the statement in case $\rho = 1$.

The statement for general ρ can clearly be reduced to the case $\rho = 1$, by considering $g(z) = f(\rho z)$.

Then $g(z) = O(\rho^\alpha(1-z)^\alpha)$ and hence

$$f^{(n)}(z) = \rho^{-n} g^{(n)}(z) = \rho^{-n} O\left(\frac{n!}{n^{\alpha+1}}\right).$$

□

Recall the behaviour

$$A_n(z) = A(\rho_n) - c_n\sqrt{\rho_n - z} + d_n(\rho - z) + O((\rho - z)^{3/2})$$

exhibited by each of the generating functions for $A_n(z)$, $2 \leq n \leq \infty$.

The Transfer Theorem will allow us to handle the big Oh-term in this expansion by transforming it to an error term for the n-th MacLaurin coefficient of $A_n(z)$ of the form

$$O\left(\frac{\rho^{-k}}{k^{\frac{5}{2}}}\right)$$

as $k \to \infty$.

It remains to estimate the contributions of the main term $\sqrt{\rho - z}$ (we clearly don't have to worry about the contributions of the linear term). By the generalised binomial theorem, the contributions of these terms to the MacLaurin coefficients of $A(z)$ are given by

$$\sqrt{\rho - z} = \sqrt{\rho}\sqrt{1 - \frac{z}{\rho}} = \sqrt{\rho}\sum_{k=0}^{+\infty}\binom{\frac{1}{2}}{k}\rho^{-k}(-z)^k.$$

To estimate the terms

$$\binom{\frac{1}{2}}{k}(-1)^k = \binom{k - \frac{3}{2}}{k} = \frac{\Gamma(n - \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)},$$

we use the following version of Stirling's approximation formula.

Lemma 3.7

For $s \in \mathbb{R}_{>0}$:

$$\Gamma(s) = \sqrt{\frac{2\pi}{s}}\left(\frac{s}{e}\right)^s\left(1 + O\left(\frac{1}{s}\right)\right)$$

as $s \to +\infty$.

Proof

We deduce this from the formula

$$\Gamma(s) = \lim_{N\to\infty}\frac{N^s N!}{s(s+1)\dots(s+N)},$$

by Euler MacLaurin summation.
Taking logarithms of both sides yields

$$\log \Gamma(s) = \lim_{N\to\infty} \left(s\log N + \sum_{k=2}^{N} \log k - \sum_{k=0}^{N} \log(s+k) \right),$$

It suffices to deduce that

$$\log \Gamma(s) = s\log s - s - \frac{1}{2}\log s + c + O\left(\frac{1}{s}\right),$$

for a constant c whose value can be determined by use of the Wallis product formula.
Applying Euler MacLaurin summation for the second summation in this limit gives:

$$\begin{aligned}\sum_{k=0}^{N} \log(s+k) = \log(s) + \int_0^N \log(s+x)dx - \left(\left(-\frac{1}{2}\right)\log(s+N) + \frac{1}{2}\log s\right) \\ + \frac{1}{2}\left(\frac{1}{6(s+N)} - \frac{1}{6s}\right) + \frac{1}{2}\int_0^N \frac{B_2(\{x\})}{(s+x)^2}dx \\ = (s+N)\log(s+N) + (1-s)\log s - N + \frac{1}{2}\log(s+N) - \frac{1}{2}\log s \\ + \frac{1}{12(s+N)} - \frac{1}{12s} + \frac{1}{2}\int_0^{+\infty} \frac{B_2(\{x\})}{(s+x)^2}dx - \frac{1}{2}\int_N^{+\infty} \frac{B_2(\{x\})}{(s+x)^2}dx.\end{aligned}$$

Here $B_2(x)$ is the second Bernoulli Polynomial. Since $|B_2(x)| \leq 1$ on $[0,1]$, we find

$$\int_N^{+\infty} \frac{B_2(\{x\})}{(s+x)^2}dx = O_s\left(\frac{1}{N}\right),$$

and

$$c(s) := \int_0^{+\infty} \frac{B_2(\{x\})}{(s+x)^2}dx = O\left(\frac{1}{s}\right).$$

In this same way, we find for $s = 1$,

$$\sum_{k=1}^{N} \log k = N(\log N - 1) + \frac{1}{2}\log N + c + O\left(\frac{1}{N}\right),$$

with $c = c(1) - \frac{1}{12}$. Putting all this together we obtain

$$\begin{aligned}\log \Gamma(s) = \lim_{N\to+\infty} \Bigg(s\log\left(\frac{N}{s+N}\right) + N\log\left(\frac{N}{s+N}\right) - N + \frac{1}{2}\log\left(\frac{N}{s+N}\right) \\ + c + O\left(\frac{1}{N}\right) - \log s + s\log s + N + \frac{1}{2}\log s - \frac{1}{12(s+N)} + \frac{1}{12s} + O\left(\frac{1}{s}\right)\Bigg) \\ = s\log s - s - \frac{1}{2}\log s + c + O\left(\frac{1}{s}\right).\end{aligned}$$

□

Remark:
With a bit more care, one can deduce that the bounds in Lemma 3.7 continue to hold uniformly for complex s in

$$\{s \in \mathbb{C} : |s| \geq \delta \text{ and } -\pi + \delta < \arg(s) < \pi - \delta\},$$

where $\delta > 0$ is fixed, see Montgomery and Vaughan (2007).

Lemma 3.8
For any $\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$,

$$\binom{n-\alpha-1}{n} = \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}\left(1 + O\left(\frac{1}{n}\right)\right).$$

Proof

$$\begin{aligned}
\binom{n-\alpha-1}{n} &= \frac{\Gamma(n-\alpha)}{\Gamma(-\alpha)\Gamma(n+1)} \\
&= \frac{\sqrt{\frac{2\pi}{n-\alpha}}\left(\frac{n-\alpha}{e}\right)^{n-\alpha}(1+O(\frac{1}{n}))}{\Gamma(-\alpha)\sqrt{\frac{2\pi}{n+1}}\left(\frac{n+1}{e}\right)^{n+1}\left(1+O\left(\frac{1}{n}\right)\right)} \\
&= \frac{1}{\Gamma(-\alpha)}\frac{e^{1+\alpha}(n-\alpha)^{n-\alpha-\frac{1}{2}}}{(n+1)^{n+\frac{1}{2}}}\left(1+O\left(\frac{1}{n}\right)\right) \\
&= \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}\frac{e^{1+\alpha}(1-\frac{\alpha}{n})^{n-\alpha-\frac{1}{2}}}{(1+\frac{1}{n})^{n+\frac{1}{2}}}\left(1+O\left(\frac{1}{n}\right)\right) \\
&= \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}e^{1+\alpha+(n-\alpha-\frac{1}{2})\log(1-\frac{\alpha}{n})-(n+\frac{1}{2})\log\left(1+\frac{1}{n}\right)}\left(1+O\left(\frac{1}{n}\right)\right) \\
&\overset{(*)}{=} \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}e^{O(\frac{1}{n})}\left(1+O\left(\frac{1}{n}\right)\right) \\
&= \frac{n^{-\alpha-1}}{\Gamma(-\alpha)}\left(1+O\left(\frac{1}{n}\right)\right),
\end{aligned}$$

where step $(*)$ follows using the mean value theorem. □

Theorem 3.9
There exist positive real constants $(b_n)_{2\leq n<\omega}$, such that $\forall n \geq 2$

$$t_n(k) = \frac{b_n}{2\sqrt{\pi k^3}}\rho_n^{-k}\left(1+O\left(\frac{1}{k}\right)\right) \quad \text{as } k \to +\infty.$$

There exists a real positive constant b such that

$$t(k) = \frac{b}{2\sqrt{\pi k^3}}\rho^{-k}\left(1+O\left(\frac{1}{k}\right)\right) \quad \text{as } k \to +\infty.$$

Proof
This follows now from the expansion

$$A_n(z) = A(\rho_n) - c_n\sqrt{\rho_n - z} + d_n(\rho_n - z) + O((\rho_n - z)^{3/2}),$$

for $2 \leq n \leq \infty$, by applying the Transfer Theorem with $\alpha = 3/2$ (this gives an error term of order $O(k^{-\frac{5}{2}})$) and using the approximation in Lemma 3.8 with $\alpha = 1/2$ (containing an error term of the same order $O(k^{-\frac{5}{2}})$). We obtain the above expression by evaluating $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$ and putting $b_n = \sqrt{\rho}\, c_n$. □

Without extra effort, we can now also obtain asymptotics for some elementary transformations of the sequences $(t(k))_k$ and $(t_n(k))_k$.

Definition 3.3
We let $t_{\leq}(k)$ denote the number of non-isomorphic Γ-trees with number of vertices not exceeding k.

It is clear that

$$t_{\leq}(k) = \sum_{l=0}^{k} t(l).$$

Hence the generating function of the sequence $(t_{\leq}(k))_k$ is given by

$$A_{\leq}(z) = \sum_{k=0}^{+\infty} z^k t_{\leq}(k) = \frac{A(z)}{1-z}.$$

Theorem 3.10
Let b be the positive real constant mentioned in Theorem 3.9, then

$$t_{\leq}(k) = \frac{b}{2(1-\rho)\sqrt{\pi k^3}} \rho_n^{-k} \left(1 + O\left(\frac{1}{k}\right)\right) \qquad \text{as } k \to +\infty.$$

Proof
From

$$A(z) = A(\rho) - c\sqrt{\rho - z} + d(\rho - z) + O((\rho - z)^{3/2}),$$

we obtain the following expansion for the generating function of the sequence $(t_{\leq}(k))_k$:

$$\frac{A(z)}{1-z} = \frac{A(\rho)}{1-\rho} - \frac{c}{1-\rho}\sqrt{\rho - z} + d'(\rho - z) + O((\rho - z)^{3/2}).$$

We readily repeat the steps from Theorem 3.9 to derive the result from the Transfer Theorem and Lemma 3.8. The constant b is again equal to $\sqrt{\rho}\, c$. □

3.4 Convergence of the radii of convergence

From the trivial property

$$\frac{A_k(x)}{x} \geq 1 \qquad \forall x \in \mathbb{R}_{>0}, 1 \leq k \leq +\infty,$$

we can deduce the following property in a very elementary way (and thus justify the notation $\rho_\infty = \rho$).

Lemma 3.9
$\rho_k \to \rho$ as $k \to \infty$.

Proof
Suppose $\rho_k \not\to \rho$. Clearly $\rho \leq \ldots \leq \rho_k \leq \rho_{k-1} \leq \ldots \leq \rho_1$.
Hence $\rho_k \to \rho' > \rho$.

Then $A(\rho')$ diverges, so in particular $\sup_l \sum_{k=1}^{l} t(k){\rho'}^k > 1$. But

$$\begin{aligned}\sup_l \sum_{k=1}^{l} t(k){\rho'}^k &\leq \sup_l \sum_{k=1}^{l} t_{l'}(k){\rho'}^k \\ &\leq \sup_l \sum_{k=1}^{l} t_{l'}(k)\rho_{l'+1}^k \\ &\leq \sup_l A_{l'}(\rho_{l'+1}) \\ &= \sup_l \frac{\rho_{l'+1}}{A_{l'+1}(\rho_{l'+1})} \leq 1\end{aligned}$$

where $l'(l) < \omega$ is such that $\forall k \leq l\ t_{l'}(k) = t(k)$. A contradiction. □

Summing up, we have proved the following.

Theorem 3.11
There exist $q, b \in \mathbb{R}_{>0}$ and two sequences $(b_n)_{n\in\omega}$, $(q_n)_{n\in\omega}$ of positive reals, $(q_n)_{n\in\omega}$ increasing, such that:

- $t(k) \sim b\frac{q^k}{k^{\frac{3}{2}}}$ as $k \to \infty$.
- $t_{\leq}(k) \sim \frac{qb}{q-1}\frac{q^k}{k^{\frac{3}{2}}}$ as $k \to \infty$.
- $t_n(k) \sim b_n\frac{q_n^k}{k^{\frac{3}{2}}}$ as $k \to \infty$.
- $q_n \to q$ as $n \to \infty$.

References and Remarks

The proof of Theorem 3.1 follows a well known general pattern, see Harary et al. (1975), Drmota (2004), Bell et al. (2006) or Flajolet and Sedgewick (2009). The proof of the Weierstraß Preparation Theorem can be found in Markushevich (1985). The Transfer Theorem is due to Flajolet and Sedgewick (2009).
We thank Michael Drmota for helpful comments on the proof of Theorem 3.6.
One can use the method outlined on page 477 of Flajolet and Sedgewick (2009) to explicitly compute the constant ρ, by help of the relation 3.9. Using the Sage-system (and without any optimization), we found in this way to fifteen decimals accurate $\rho = 0.177099522303285\ldots$. This agrees with the decimal expansion of the constant $q = \rho^{-1}$ which is available in the OEIS up to 61 decimals (see Sloane (2018a)). One can use the value of ρ to explicitly compute b.

A possible further line of study entails determining further analytic properties of the analytic functions $A_n(z)$ and $A(z)$. One might for example study the rate at which $\rho_n \to \rho$.

4

Additional applications in enumerative graph theory

We illustrate the combinatorial value of the results on Γ-trees discussed in the previous chapters, by deriving results on the asymptotic behaviour of certain classic families of graphs.

4.1 Matched rooted trees

One of the classical notions in graph theory is that of a matching.

Definition 4.1
A *matching* of a graph G is a set of disjoint edges of G.
A matching is *perfect* if any vertex of G is contained in an edge of this matching.
A graph G is *matched* if there exists a perfect matching for G.

We will now study matched rooted trees, in the combinatoric literature also known as "trees with 1-factors". The asymptotics of such trees are the main subject of study in e.g. the article Simion (1991), some of whose main results we will retrieve, by aid of previous work on Γ-trees.
Clearly a graph of odd order is never matched. A rooted tree of even order can be either matched (e.g. the trees in Figure 4.1) or not matched (e.g. the trees in Figure 4.2).

Lemma 4.1 If a forest τ is matched, it has a unique perfect matching.

Proof
By induction on the order n of τ.
Clearly, the statement is true when $n \in \{0, 1\}$.
In the induction step, suppose τ has a perfect matching M.
We can assume τ is a tree, for otherwise, it follows at once that τ has a unique perfect matching by applying the induction hypothesis on each of the connected

Figure 4.1: Examples of matched trees

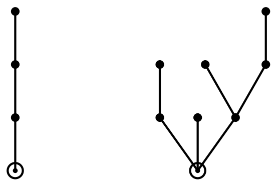
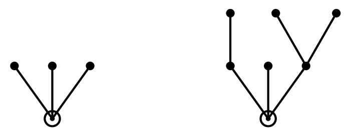


Figure 4.2: Examples of trees that are not matched



components of τ. Since τ is a tree, it certainly has a vertex a that is a leaf. Then a is connected to a unique vertex b, and we should have $\{a, b\} \in M$. Then $M \setminus \{a, b\}$ is a perfect matching for the forest obtained from τ by deleting the vertices a and b. By the induction hypothesis, $M \setminus \{a, b\}$ and then also M are uniquely determined. $\square$

The numbers of non-isomorphic planted matched rooted trees on n vertices are given by

$$0, 0, 1, 0, 1, 0, 3, 0, 10, 0, 39, \ldots.$$

So the numbers $t(n)$ come up again.

Theorem 4.1
The number of planted matched rooted trees on $2n$ vertices is equal to the number of Γ-trees on n vertices.

Proof
We adjoin to any planted matched rooted tree τ of order $2n$ a Γ-tree τ' of order n in the following way.

Let $M = \{e_1, \ldots, e_n\}$ be the unique perfect matching for τ. Let τ' be the tree with vertices $e_1, \ldots, e_n$, where e_i and e_j are adjacent in τ', exactly when τ contains an edge connecting one of the vertices in e_i with one of the vertices in e_j.

Note that τ' is indeed a tree: from a cycle $(e_{k_1}, \ldots, e_{k_n})$ in τ', we can construct a cycle in τ in the following way.
Let $v_1 \in e_{k_1}$, let v_2 be a vertex in e_{k_2} that is adjacent (in τ) to v_1, if there is such a vertex. Otherwise, let v_2 be the unique vertex in $e_{k_1} \setminus \{v_1\}$. In this case, there exists v_3 in e_{k_2} that is adjacent to v_2.
Continuing in this fashion, we construct a sequence of successively adjacent vertices of τ: $(v_1, v_2, \ldots, v_r)$, with v_r again in e_{k_1}. It follows that either $v_r = v_1$ or v_r and v_1 are connected by e_{k_1}. Hence, we have found a cycle in τ.

We make τ' into a Γ-tree: the root of τ' is the unique $e_i \in M$ containing the root of τ.
Next, we give a direction to the edges in τ', not adjacent to its root.
Let e_1, e_2 be two edges of τ (hence vertices of τ'), not containing the root of τ, that are adjacent in the tree τ'. Without loss of generality, e_1 contains the vertex v_1 of τ that is closest (has least distance) to the root of τ. There are then two structurally different possibilities: either v_1 is adjacent to a vertex of e_2, or e_2 is connected to e_1 via its other vertex.

In the first case (see Figure 4.3) we endow the edge in τ' connecting e_1 and e_2 with the direction from e_2 to e_1. In the second case (see Figure 4.4), we give it the direction from e_1 to e_2.

It is obvious how this construction can be reversed, hence $\tau \mapsto \tau'$ is a bijection between the rooted matched planted trees of order $2n$ and the Γ-trees of order n.
□

Figure 4.3: Case 1: direction from e_2 to e_1

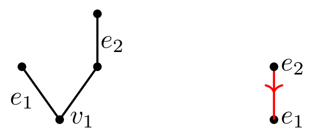


Figure 4.4: Case 2: direction from e_1 to e_2

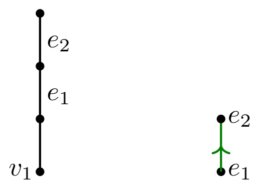
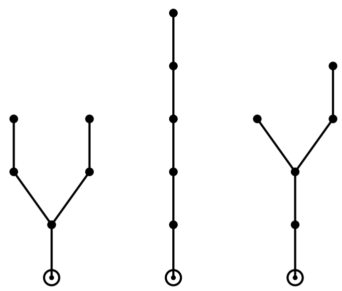


Figure 4.5: The planted matched trees with $2n$ vertices, in the same order as the corresponding Γ-trees of Figure 2.6

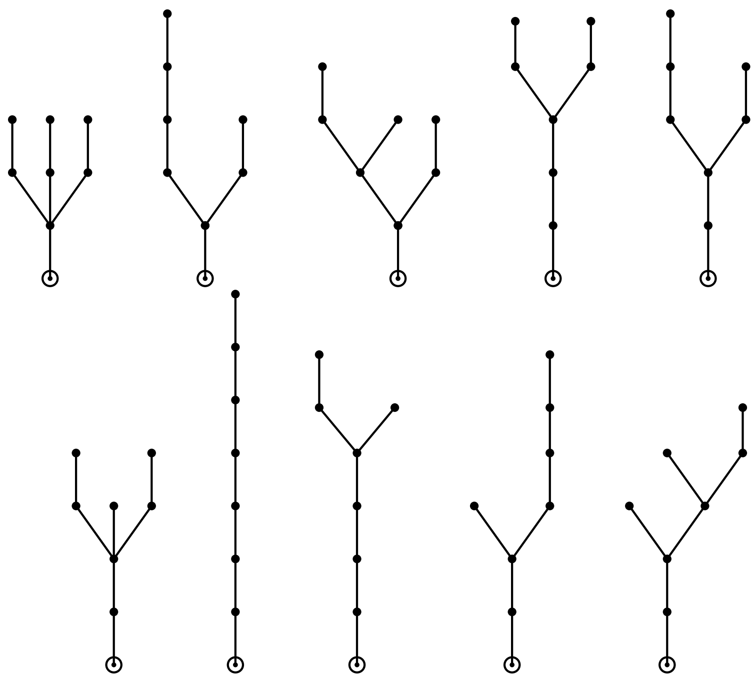
2 vertices:

4 vertices:

6 vertices:



8 vertices:

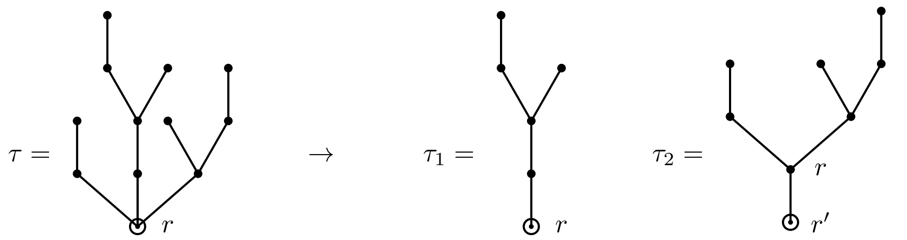


From *planted* matched trees it is a small step towards counting all matched trees.

Theorem 4.2
The number of matched rooted trees on $2n$ vertices equals the number of pairs (τ_1, τ_2) where τ_1, τ_2 are Γ-trees with $|\tau_1| + |\tau_2| = n + 1$.

Proof
Using Theorem 4.1, we see that we have to prove that the number of matched rooted trees on $2n$ vertices equals the number of pairs (τ_1, τ_2) where τ_1, τ_2 are planted matched rooted trees with $|\tau_1|+|\tau_2| = 2n+2$. Let τ be a matched rooted tree with order $2n$ and root r. We describe the pair (τ_1, τ_2) of planted matched rooted trees that corresponds to τ. Let M be the unique perfect matching of τ. Then there is a unique vertex m of τ such that $\{m, r\} \in M$. Deleting the root from τ, we obtain a forest with connected components $\sigma_1, \ldots, \sigma_l$. Without loss of generality, $m \in \sigma_1$. Then τ_1 is the rooted subtree of τ determined by the root r and the vertices in σ_1. Let σ be the subtree of τ obtained by deleting the vertices in σ_1 from τ. This tree is not matched. The matched rooted subtree τ_2 is obtained by adding a new vertex r' and adding the single edge $\{r, r'\}$. It has root r'. This construction is illustrated in Figure 4.6.
Again, the construction is clearly reversible. $\square$

Figure 4.6: Illustration of the construction in the proof of Theorem 4.2



4.2 The family of 2-trees

The Γ-trees are also directly related to so-called *2-dimensional trees* or *2-trees*. We think of these as connected graphs consisting of triangles in such a way that there is no cycle of triangles. Also this family did not remain without attention in the graph theoretic literature, 2-trees have for example been studied in Fowler et al. (2002) and Labelle et al. (2002).

Definition 4.2
The family of 2-trees is a family of graphs that is inductively defined as follows:

$\kappa_2 =$  is a 2-tree, if γ is a graph containing vertices a, b, c, such that:

- $\deg(a) = 2$,
- a is adjacent to both b and c,
- b and c are adjacent,
- the graph obtained by deleting the vertex a from γ is a 2-tree,

then γ is also a 2-tree.

A rooted 2-tree is a 2-tree γ with a privileged edge (the root-edge).
An oriented rooted 2-tree is a rooted 2-tree together with a direction assigned to the root-edge.
The set of cells of a rooted 2-tree γ consists of the root-edge of γ together with the triangles of γ.
Two oriented rooted 2-trees γ_1, γ_2 are isomorphic if there exists a graph isomorphism from γ_1 to γ_2 that sends the root-edge of γ_1 to the root-edge of γ_2 and preserves its direction.

Note that given the direction of the root-edge of a 2-tree γ, there is a unique way of making γ into a directed graph such that any triangle of γ is either clockwise or anti-clockwise oriented by its edges.

Lemma 4.2
The number of oriented rooted 2-trees with n triangles equals the number of Γ-trees on $n+1$ vertices.

Proof
We adjoin inductively, following the inductive definition of 2-trees, to any oriented rooted 2-tree γ, a Γ-tree $C(\gamma)$, together with a bijection f_γ between the cells of γ and the vertices of $C(\gamma)$.

We make the unique oriented 2-tree  correspond to the unique Γ-tree of order 1, namely $\odot$. In this case, b_γ sends the unique cell of γ (the root-edge) to the unique vertex of $\odot$.
Next, let γ be an oriented 2-tree with a vertex a as in Definition 4.2 and suppose that we have already defined the Γ-tree $C(\gamma \setminus \{a\})$ corresponding to $\gamma \setminus \{a\}$, together with a bijection $f = f_{\gamma \setminus \{a\}}$ between the cells of $\gamma \setminus \{a\}$ and the vertices of $C(\gamma \setminus \{a\})$. By the previous remark, we can assume every edge of γ to be directed in such a way that any triangle of γ is either clockwise or anti-clockwise oriented by its edges.We adjoin to $C(\gamma \setminus \{a\})$ a new vertex v_1. Let v_2 be the vertex in

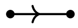
$$\{v \in C(\gamma \setminus \{a\}) : f^{-1}(v) \text{ contains the edge } \{\text{b,c}\}\}$$

that is closest to the root of $C(\gamma \setminus \{a\})$. It will follow by induction that for all adjacent v_1, v_2 in $C(\gamma \setminus \{a\})$, the cells $f^{-1}(v_1)$ and $f^{-1}(v_2)$ have an edge in common. Hence the set considered will always have a unique element closest to the root, for otherwise a cycle of triangles would be induced in γ.
We add to $C(\gamma \setminus \{a\})$ the edge $\{v_1, v_2\}$.
In case $f^{-1}(v_2)$ is the root-edge, we do not need to colour $\{v_1, v_2\}$.
If $f^{-1}(v_2)$ is a triangle, let e be the edge in γ that $f^{-1}(v_2)$ shares with the cell corresponding via f to the direct successor of v_2 on the path from v_2 to the root in $C(\gamma \setminus \{a\})$. If the edge e is oriented towards the edge $\{b, c\}$, we define the

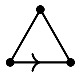
direction of $\{v_1, v_2\}$ to be upward, otherwise we define it to be downward. This construction can clearly be reversed to find for any Γ-tree a corresponding oriented, rooted 2-tree (this time with induction on Γ-trees), moreover, it is clear that both correspondences will carry over isomorphisms of oriented rooted 2-trees, respectively Γ-trees. □

Figure 4.7: The oriented rooted 2-trees with n triangles, in the same order as the corresponding Γ-trees of Figure 2.6

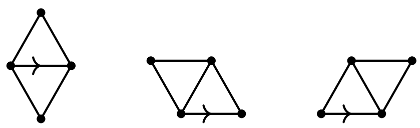
0 triangles:



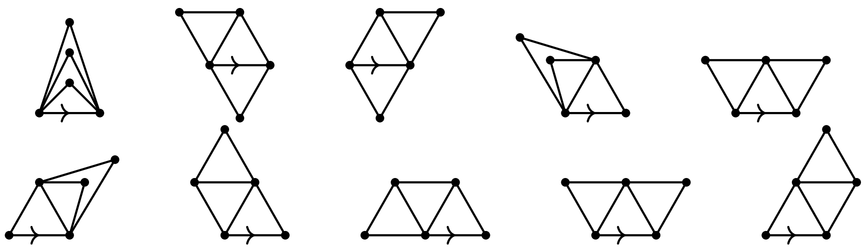
1 triangle:



2 triangles:



3 triangles:



4.3 Asymptotics

We obtain now in one strike three main results of Simion (1991) and Fowler et al. (2002).

Theorem 4.3
Let m_{2n}, p_{2n} and a_n be the number of matched rooted trees on $2n$ vertices, the number of planted matched rooted trees on $2n$ vertices and the number of

oriented rooted trees on $n+1$ vertices, respectively. Then

$$m_{2n} = \frac{b}{\sqrt{2\pi n^3}}\rho^{-(n+\frac{1}{2})}\left(1+O\left(\frac{1}{n}\right)\right)$$
$$p_{2n} = \frac{b}{2\sqrt{\pi n^3}}\rho^{-n}\left(1+O\left(\frac{1}{n}\right)\right)$$
$$a_n = \frac{b}{2\sqrt{\pi n^3}}\rho^{-n}\left(1+O\left(\frac{1}{n}\right)\right)$$

where ρ and b are the same constants as in Theorem 3.9.

Proof
Let $M(z)$, $P(z)$ and $G(z)$ be the corresponding generating functions. From Theorem 4.1 and Theorem 4.2 we have $P(z) = G(z) = A(z)$ and $M(z) = \frac{A^2(z)}{z}$. The last two equations follow immediately from Theorem 3.9. From

$$A(z) = A(\rho) - c\sqrt{\rho - z} + d(\rho - z) + O((\rho - z)^{3/2}),$$

we find

$$M(z) = M(\rho) - c\sqrt{\frac{2}{\rho}}\sqrt{\rho - z} + d'(\rho - z) + O((\rho - z)^{3/2}).$$

Using the method outlined in Section 3.3, this leads directly to

$$m_{2n} = \sqrt{\frac{2}{\rho}}\frac{b}{2\sqrt{\pi n^3}}\rho^{-n}\left(1+O\left(\frac{1}{n}\right)\right),$$

where in line with our previous definitions, $b = \sqrt{\rho}\, c$. □

Using the numerical values of Otter's constant and ρ available in the OEIS, we find $\rho_2^2 < \rho$, this leads to the following corollary.

Corollary 4.1
Let X_n be the stochastic variable assuming 1 when a unformly random chosen rooted tree on $2n$ vertices is matched and 0 otherwise.
Then $P(X_n = 1)$ converges to 0 as $n \to +\infty$ exponentially fast:

$$P(X_n = 1) = O\left(\left(\frac{\rho_2^2}{\rho}\right)^n\right).$$

References and Remarks

The combinatorial correspondence between 2-trees and the ordinals below Γ_0 was noted by Andreas Weiermann.
The combinatorial correspondences between Γ-trees, matched trees and 2-trees are mentioned on the OEIS page of sequence A005750 (Sloane (2018a)), there is no reference to be found here to the ordinals below Γ_0.

5

A Phase transition for Sim

5.1 The ordinal Γ_0

We now relate the Γ-trees to the ordinals smaller than a specific countable ordinal Γ_0, to be introduced in Definition 5.3. Since there are countably many objects of both kinds, it is of course of no surprise that such a bijection exists. It is however surprising that we can establish an order isomorphism.

In this section, we work in ZFC set theory. We endow any ordinal α with its order topology $\tau_<$, i.e. the topology with subbasis

$$\mathscr{S} = \{[0,\beta[,]\beta,\alpha[: \beta < \alpha\}.$$

Lemma 5.1
Let α be an ordinal, $\tau_<$ its order topology, then

(i) $(\alpha, \tau_<)$ is a Hausdorff topological space,

(ii) for any successor ordinal $\beta + 1$ in α, $\{\beta + 1\}$ is clopen,

(iii) for any limit ordinal $\gamma \neq 0$ in α,

$$B_\gamma = \{]\beta,\gamma] : \beta < \gamma\}$$

is a local τ-base for γ.

Proof

(i) For $\beta_1 < \beta_2 < \alpha$, we have $\beta_1 \in [0, \beta_1 + 1[$, $\beta_2 \in]\beta_1, \alpha[$.
$[0, \beta_1 + 1[$, $]\beta_1, \alpha[$ are open and have empty intersections.

(ii) $\{\beta + 1\} =]\beta, \beta + 2[$ is open. In a Hausdorff space, any singleton is closed.

(iii) $]\beta,\gamma] =]\beta,\alpha[\cap [0,\gamma+1[$ is open.
Because γ is a limit ordinal, we have $\gamma \in [\beta_1,\beta_2[\Rightarrow \gamma \in [\beta_1+1,\beta_2[$. Hence, for any set U in $\mathscr{S}$ that contains γ, there exists a set V in B_γ such that $\gamma \in V \subseteq U$. Since B_γ is closed under finite intersections, for any $\tau_<$-open set $U_1 \cap \ldots \cap U_k$ that contains γ, we find $V_1 \cap \ldots \cap V_k \in B_\gamma$ containing γ, with

$$V_1 \cap \ldots \cap V_k \subseteq U_1 \cap \ldots \cap U_k.$$

□

Lemma 5.2
For any ordinal α and $A \subseteq \alpha$, $\overline{A} \setminus A$ consists of the limit ordinals γ in $\alpha \setminus A$ with the property $\sup A \cap \gamma = \gamma$.

Proof
$\overline{A} \setminus A$ can not contain successor ordinals because any successor ordinal is isolated. The limit ordinal γ in $\alpha \setminus A$ will be in $\overline{A} \setminus A$ precisely when for any $\beta < \gamma$, there is $\delta > \beta$ in $A \cap \gamma$. □

Corollary 5.1
$A \subseteq \alpha$ is closed in the order topology $\tau_<$ on α precisely when
A contains any limit ordinal $\gamma < \alpha$ with the property $\sup A \cap \gamma = \gamma$.

Definition 5.1
A subset S of a cardinal κ is called (a) *club* if it is both closed and unbounded in κ.

Lemma 5.3
Let κ be a cardinal with uncountable cofinality and let $\lambda < \mathrm{cf}(\kappa)$.
If $(C_\alpha)_{\alpha<\lambda}$ is a family of club sets in κ, then $\bigcap_{\alpha<\lambda} C_\alpha$ is club in κ.

Proof
$\bigcap_{\alpha<\lambda} C_\alpha$ is clearly closed, as the intersection of closed sets.
For $\alpha_0 < \kappa$ arbitrarily, construct by recursion on the well-order $\omega \times \lambda$ the function $\Theta : \omega \times \lambda \to \kappa$ as follows:

$$\Theta(0,0) = \alpha_0 + 1,$$
$$(\forall (n,\alpha) \in \omega \times \lambda)\,(\Theta(n,\alpha) = \min C_\alpha \setminus \sup_{(m,\beta)<(n,\alpha)} \Theta(m,\beta) + 1).$$

Let $\sigma = \sup_{\substack{n<\omega \\ \alpha<\lambda}} \Theta(n,\alpha)$. Then $\kappa > \sigma > \alpha_0$ and for any $\alpha < \kappa$,

$$\sigma = \sup_{n<\omega} \Theta(n,\alpha) \in C_\alpha.$$

□

Let κ be a regular cardinal and $A \subseteq \kappa$ unbounded.
The function $\mathrm{Enum}(A) : \kappa \to A$ is the unique order isomorphism between the well-orders κ and A.

Definition 5.2
A function $f : \alpha \to \beta$ is *normal* when it is continuous and (strictly) increasing.

Lemma 5.4
An increasing function $f : \alpha \to \beta$ is continuous if and only if for any $S \subset \alpha$, that is bounded in α, $\sup f[S] = f(\sup S)$.

Proof
$\boxed{\Leftarrow}$ Let $\gamma \in \alpha$ a limit ordinal. Then $f(\gamma) = \sup f[\gamma]$ is a limit ordinal. From $f(\gamma) = \sup f[\gamma]$, we also infer that for any $\delta \in f(\gamma)$, there is $\delta' \in \gamma$ such that $\delta < f(\delta') < f(\gamma)$. This implies that $f^{-1}(]\delta, f(\gamma)]) \supseteq]\delta', \gamma]$ and we can conclude that f is continuous.
$\boxed{\Rightarrow}$ If $\sup S \in S$, this is trivial. Else, $\gamma := \sup S$ is a non-zero limit ordinal and therefore not isolated. Because f is continuous, this implies that $f(\gamma)$ is non-isolated in β and is therefore a limit ordinal.
By continuity, for any $\delta \in f(\gamma)$, there is $\delta' \in \gamma$ such that $f(]\delta', \gamma]) \subseteq]\delta, f(\gamma)]$.
Hence, $\sup f[S] = \sup f[\gamma] = f(\gamma) = f(\sup S)$. □

Lemma 5.5
Let κ be a regular cardinal and $A \subseteq \kappa$ unbounded.
Then $\mathrm{Enum}(A) : \kappa \to A$ is continuous $\iff$ A is closed in κ.

Proof
$\boxed{\Rightarrow}$ Write $f = \mathrm{Enum}(A)$. Suppose first that f is continuous. It follows (Lemma 5.4) that for any bounded $S \subseteq \kappa$, $\sup f[S] = f(\sup S)$.
Let γ be a limit ordinal in κ with $\sup \gamma \cap A = \gamma$. Then there is a bounded $S \subseteq \kappa$ with $f[S] = \gamma \cap A$ and this implies

$$A \ni f(\sup S) = \sup f[S] = \sup \gamma \cap A = \gamma.$$

$\boxed{\Leftarrow}$ Suppose A is closed in κ.
We check that for any bounded $S \subset \kappa$, $\sup f[S] = f(\sup S)$.
This is trivial if $\sup S \in S$.
If $\sup S \notin S$, then at least $\sup f[S] \in A$, since A is closed.
Hence, $\sup f[S] = f(\gamma)$ for $\gamma < \kappa$.
Since f is increasing, both $\gamma < \sup S$ and $\gamma > \sup S$ are impossible, hence $\gamma = \sup S$. □

Lemma 5.6
Let $\kappa > \omega$ be regular and $f : \kappa \to \kappa$ normal. Then the set of fixpoints of f, $\mathrm{Fix}(f)$, is a club set in κ.

Proof
$\mathrm{Fix}(f)$ is closed because $\tau_<$ is Hausdorff.
Let $\alpha_0 < \kappa$ be arbitrary. Choose β_0 such that $f(\beta_0) > \alpha$.
Define $(\beta_n)_{n<\omega}$ by $\beta_{n+1} = f(\beta_n)$.
Then

$$f(\sup_{n<\omega} \beta_n) = \sup_{n<\omega} f(\beta_n) = \sup_{n<\omega} \beta_n$$

is a fixpoint of f larger than α. □

We can now define the Feferman-Schütte ordinal Γ_0.
As before, we use the notation ω_1 for the smallest uncountable ordinal.

Definition 5.3
Let $\varphi_0 : \omega_1 \to \omega_1$ be the enumerating function of the additively indecomposable ordinals smaller than ω_1, i.e. $(\forall \alpha \in \omega_1)\varphi_0(\alpha) = \omega^\alpha$.
Define for each $\alpha < \omega_1$ the normal function $\varphi_\alpha : \omega_1 \to \omega_1$ by

$$\varphi_\alpha = \text{Enum}\left(\bigcap_{\beta<\alpha} \text{Fix}(\varphi_\beta)\right).$$

Observe that each φ_α is indeed well-defined by recursion on ω_1: the intersection $\bigcap_{\beta<\alpha} \text{Fix}(\varphi_\beta)$ is an intersection of countably many club sets (by Lemma 5.6) and hence (by Lemma 5.3) again a club. It follows that the enumeration function of $\bigcap_{\beta<\alpha} \text{Fix}(\varphi_\beta)$ is normal (Lemma 5.5) and has domain ω_1.

Definition 5.4
We define:

$$\Gamma_0 := \min\{\alpha < \omega_1 : \varphi_\alpha(0) = \alpha\}.$$

To make sure that this definition is valid, we need to prove that the set $\{\alpha < \omega_1 : \varphi_\alpha(0) = \alpha\}$ is non-empty. In Lemma 5.8, we show that this set is even unbounded.
We will need the following lemma.

Lemma 5.7
For every ordinal $\alpha < \omega_1$,

$$\varphi_\alpha(0) \geq \alpha.$$

Proof
Otherwise, there would be a minimal $\alpha < \omega_1$ with $\varphi_\alpha(0) < \alpha$. Then

$$\begin{aligned}\varphi_\alpha(0) \in \bigcap_{\beta<\alpha} \text{Fix}(\varphi_\beta) &\Rightarrow \varphi_\alpha(0) \in \text{Fix}(\varphi_{\varphi_\alpha(0)})\\ &\Rightarrow \varphi_\alpha(0) = \varphi_{\varphi_\alpha(0)}(\varphi_\alpha(0)) > \varphi_{\varphi_\alpha(0)}(0)\end{aligned}$$

but this is in contradiction with minimality of α. □

Lemma 5.8
$S = \{\alpha < \omega_1 : \varphi_\alpha(0) = \alpha\}$ is club.

Proof
S is closed.
Let γ be a countable limit ordinal and suppose $\sup S \cap \gamma = \gamma$.
We have to prove that $\gamma \in \bigcap_{\beta<\gamma} \text{Fix}(\varphi_\beta)$.
Let $\beta < \gamma$. Then $\sup[(S \cap \gamma) \setminus (\beta + 1)] = \gamma$ and $(S \cap \gamma) \setminus (\beta + 1) \subseteq \text{Fix}(\varphi_\beta)$.
So,

$$\begin{aligned}\varphi_\beta(\gamma) &= \varphi_\beta(\sup[(S \cap \gamma) \setminus (\beta + 1)])\\ &= \sup \varphi_\beta[(S \cap \gamma) \setminus (\beta + 1)]\\ &= \sup(S \cap \gamma) \setminus (\beta + 1)\\ &= \gamma.\end{aligned}$$

$\boxed{S \text{ is unbounded.}}$
Consider $f : \omega_1 \to \omega_1 : \alpha \mapsto \varphi_\alpha(0)$.
Let $\alpha_0 < \kappa$. Define $(\alpha_n)_{n<\omega}$ by $\alpha_{n+1} = \varphi_{\alpha_n}(0) + 1$. This sequence is increasing. Define $\sigma = \sup\limits_{n<\omega} \alpha_n$.
Let $\beta < \sigma$ arbitrary. Then

$$\varphi_\beta(\sigma) = \varphi_\beta(\sup_{n<\omega} \alpha_n) = \sup_{\substack{n<\omega \\ \alpha_n > \beta}} \varphi_\beta(\alpha_n) = \sup_{\substack{n<\omega \\ \alpha_n > \beta}} \alpha_n = \sigma.$$

Hence $\varphi_\sigma(0) = \sigma$. □

Lemma 5.9
For ordinals $\alpha_1, \beta_1, \alpha_2, \beta_2 < \omega_1$, $\varphi_{\alpha_1}(\beta_1) = \varphi_{\alpha_2}(\beta_2)$ holds precisely when one of the following holds:

$$\begin{array}{lll} \alpha_1 < \alpha_2 & \text{and} & \beta_1 = \varphi_{\alpha_2}(\beta_2), \\ \alpha_1 = \alpha_2 & \text{and} & \beta_1 = \beta_2, \\ \alpha_1 > \alpha_2 & \text{and} & \varphi_{\alpha_1}(\beta_1) = \beta_2. \end{array}$$

Proof
Since φ_{α_1} is strictly increasing, it is clear that $\varphi_{\alpha_1}(\beta_1) = \varphi_{\alpha_1}(\beta_2)$ if and only if $\beta_1 = \beta_2$.
Suppose now $\alpha_1 < \alpha_2$, $\varphi_{\alpha_2}(\beta_2)$ is a fixpoint of all φ_γ, $\gamma < \alpha_2$ and by consequence $\varphi_{\alpha_2}(\beta_2) = \varphi_{\alpha_1}(\varphi_{\alpha_2}(\beta_2))$.
Again, since φ_{α_1} is strictly increasing, it follows that

$$\varphi_{\alpha_1}(\beta_1) = \varphi_{\alpha_2}(\beta_2) = \varphi_{\alpha_1}(\varphi_{\alpha_2}(\beta_2)) \iff \beta_1 = \varphi_{\alpha_2}(\beta_2).$$

The third case is symmetric. □

In exactly the same way, we prove the following lemma, which is very much reminiscent of Definition 2.11, defining the order relation on Γ-trees.

Lemma 5.10
For ordinals $\alpha_1, \beta_1, \alpha_2, \beta_2 < \omega_1$, $\varphi_{\alpha_1}(\beta_1) < \varphi_{\alpha_2}(\beta_2)$ holds precisely when one of the following holds:

$$\begin{array}{lll} \alpha_1 < \alpha_2 & \text{and} & \beta_1 < \varphi_{\alpha_2}(\beta_2), \\ \alpha_1 = \alpha_2 & \text{and} & \beta_1 < \beta_2, \\ \alpha_1 > \alpha_2 & \text{and} & \varphi_{\alpha_1}(\beta_1) < \beta_2. \end{array}$$

Lemma 5.11
Every additively indecomposable ordinal $\alpha < \omega_1$ can be written in a unique way in the form $\varphi_\beta(\gamma)$, with $\gamma < \alpha$ and $\beta < \omega_1$.

Proof
Unicity follows using Lemma 5.9. Indeed, if $\varphi_{\beta_1}(\gamma_1) = \alpha = \varphi_{\beta_2}(\gamma_2)$ and $\beta_1 \neq \beta_2$, then according to this lemma we have either $\alpha = \gamma_1$ or $\alpha = \gamma_2$. On the other hand, if $\beta_1 = \beta_2$ and $\varphi_{\beta_1}(\gamma_1) = \varphi_{\beta_2}(\gamma_2)$, $\gamma_1 = \gamma_2$ should hold because φ_{β_1} is strictly increasing.
Next, we prove existence of β and γ.

Let $\beta = \min\{\delta < \omega_1 : \alpha \notin \mathrm{Fix}(\varphi_\delta)\}$.
Note that $\alpha > 0$.
Since $\varphi_\alpha(\alpha) > \varphi_\alpha(0) \geq \alpha$, it follows that $\alpha \notin \mathrm{Fix}(\varphi_\alpha)$ and by consequence β is well-defined.
If $\beta = 0$, we have $\alpha = \varphi_\beta(\gamma)$ for certain $\gamma < \omega_1$, because α is additively indecomposable. If we had $\gamma \geq \alpha$, it would follow that

$$\alpha \leq \varphi_0(\alpha) \leq \varphi_0(\gamma) = \alpha$$

(where the second inequality follows because φ_0 is an increasing function), and hence $\alpha \in \mathrm{Fix}(\varphi_0)$, which is in contradiction with the definition of β.
If $\beta > 0$, α is a fixpoint of any φ_δ, $\delta < \beta$ and it follows again that there is $\gamma < \omega_1$ such that $\alpha = \varphi_\beta(\gamma)$.
If we had $\gamma \geq \alpha$, it would follow that

$$\alpha \leq \varphi_\beta(\alpha) \leq \varphi_\beta(\gamma) = \alpha,$$

and hence $\alpha \in \mathrm{Fix}(\varphi_\beta)$ which is in contradiction with the definition of β. □

Next we define $B : \omega_1 \times \omega_1 \to \varphi_0[\omega_1]$ in the following way:

$B(\beta,\gamma) = \varphi_\beta(\gamma)$ if $(\forall n < \omega)(\forall \gamma_0 \in \mathrm{Fix}(\varphi_\beta))(\gamma_0 + n \neq \gamma)$
and
$B(\beta,\gamma) = \varphi_\beta(\gamma_0 + n + 1)$ if $(\exists n < \omega)(\exists \gamma_0 \in \mathrm{Fix}(\varphi_\beta))(\gamma_0 + n = \gamma)$.

Lemma 5.12
For all ordinals $\alpha, \beta, \gamma < \omega_1$, if $B(\alpha, \beta) = \varphi_\alpha(\gamma)$, then $\gamma < \varphi_\alpha(\gamma)$.

Proof
Case I $(\forall n < \omega)(\forall \beta_0 \in \mathrm{Fix}(\varphi_\alpha))(\beta_0 + n \neq \beta)$.
Then $B(\alpha, \beta) = \varphi_\alpha(\beta)$.
Then $\varphi_\alpha(\gamma) = \varphi_\alpha(\beta) > \beta = \gamma$, follows from $\beta \notin \mathrm{Fix}(\varphi_\alpha)$.
Case II $(\exists n < \omega)(\exists \beta_0 \in \mathrm{Fix}(\varphi_\alpha))(\beta_0 + n = \beta)$.
Then $B(\alpha, \beta) = \varphi_\alpha(\beta_0 + n + 1)$ and $\gamma = \beta + 1$.
Then $\beta \leq \varphi_\alpha(\beta) < \varphi_\alpha(\gamma)$. Hence $\beta < \varphi_\alpha(\gamma)$.
Since $\varphi_\alpha(\gamma)$ is a limit ordinal, it follows that also $\gamma = \beta + 1 < \varphi_\alpha(\gamma)$. □

Lemma 5.13
For all ordinals $\alpha, \beta_1, \beta_2 < \omega_1$, if $\beta_1 < \beta_2$, then $B(\alpha, \beta_1) < B(\alpha, \beta_2)$.

Proof
Case I $(\forall n < \omega)(\forall \beta_0 \in \mathrm{Fix}(\varphi_\alpha))(\beta_0 + n \neq \beta_1)$.
Then $B(\alpha, \beta_1) = \varphi_\alpha(\beta_1) < B(\alpha, \beta_2) \in \{\varphi_\alpha(\beta_2), \varphi_\alpha(\beta_2 + 1)\}$.
Case II $(\exists n < \omega)(\exists \beta_0 \in \mathrm{Fix}(\varphi_\alpha))(\beta_0 + n = \beta_1)$.
Then $B(\alpha, \beta_1) = \varphi_\alpha(\beta_1 + 1)$.
If $\beta_2 = \beta_1 + 1$, $B(\alpha, \beta_1) = \varphi_\alpha(\beta_1 + 1) < \varphi_\alpha(\beta_2 + 1) = B(\alpha, \beta_2)$.
Else, $\beta_2 > \beta_1 + 1$, so $B(\alpha, \beta_1) = \varphi_\alpha(\beta_1 + 1) < \varphi_\alpha(\beta_2) \leq B(\alpha, \beta_2)$. □

Lemma 5.14
$B : \Gamma_0 \times \Gamma_0 \to \varphi_0[\Gamma_0]$ is a bijection.

Proof

Surjectivity

If $\alpha \in \varphi_0[\Gamma_0]$, then by Lemma 5.11, there is $\beta < \omega_1$, $\gamma < \alpha$ with $\varphi_\beta(\gamma) = \alpha$.
Case I $(\forall n < \omega)(\forall \gamma_0 \in \mathrm{Fix}(\varphi_\beta))(\gamma_0 + n \neq \gamma)$.
Then $B(\beta, \gamma) = \varphi_\beta(\gamma) = \alpha$.
Case II $(\exists n < \omega)(\exists \gamma_0 \in \mathrm{Fix}(\varphi_\beta))(\gamma_0 + n = \gamma)$.

- Case II.1 If $n > 0$, then $B(\beta, \gamma_0 + n - 1) = \varphi_\beta(\gamma) = \alpha$.
- Case II.2 If $n = 0$, then $\gamma = \varphi_\beta(\gamma) = \alpha$, but this is in contradiction with the assumption $\gamma < \alpha$.

Because Γ_0 is a fixpoint of φ_0, it follows from $\alpha \in \varphi_0[\Gamma_0]$ that $\alpha < \Gamma_0$. Hence, $\beta, \gamma \leq \alpha < \Gamma_0$.

Injectivity

Suppose that $B(\alpha_1, \beta_1) = B(\alpha_2, \beta_2)$.
Let $B(\alpha_1, \beta_1) = \varphi_{\alpha_1}(\gamma_1)$ and $B(\alpha_2, \beta_2) = \varphi_{\alpha_2}(\gamma_2)$.
If $\alpha_1 \neq \alpha_2$, then without loss of generality $\alpha_1 < \alpha_2$, and by Lemma 5.9, it would follow that $\gamma_1 = \varphi_{\alpha_2}(\gamma_2)$. This is in contradiction with $\gamma_1 < \varphi_{\alpha_1}(\gamma_1)$. Hence, $\alpha_1 = \alpha_2$. But then it follows from Lemma 5.13 that also $\beta_1 = \beta_2$. □

Lemma 5.15
For all ordinals $\alpha, \beta < \omega_1$, if $B(\alpha, \beta) < \Gamma_0$, then both of the following statements hold:

- $\alpha < B(\alpha, \beta)$,
- $\beta < B(\alpha, \beta)$.

Proof

(1) Since $B(\alpha, \beta) < \Gamma_0$, $\{\varphi_\alpha(\beta), \varphi_\alpha(\beta + 1)\} \ni B(\alpha, \beta) < \varphi_{B(\alpha,\beta)}(0)$, we find by Lemma 5.10 that $\alpha < B(\alpha, \beta)$.

(2) Let γ be such that $B(\alpha, \beta) = \varphi_\alpha(\gamma)$.
Then, by Lemma 5.12 , $\beta \leq \gamma < \varphi_\alpha(\gamma) = B(\alpha, \beta)$.

□

We can now prove the main result of this section.

Theorem 5.1

$$(\Gamma_0, <) \cong (\Gamma_{\text{tree}}, <).$$

Proof
Define $T : \Gamma_0 \to \Gamma_{\text{tree}}$ by recursion. $T(0) = \odot$.
For any $\alpha =_{\text{CNF}} \omega^{\alpha_n} + \ldots + \omega^{\alpha_0}$, we define $T(\alpha) = \text{Combine}(T(\omega^{\alpha_n}), \ldots, T(\omega^{\alpha_0}))$.
For $\alpha \in \varphi_0[\Gamma_0]$, we define $T(\alpha) = \text{Pair}(T(\beta), T(\gamma))$, where $B(\beta, \gamma) = \alpha$.

(1) T is surjective.
We prove by induction on $n < \omega$ that every Γ-tree τ with $|\tau| = n$ is in $\mathrm{Im}(T)$ and that if τ is planted, then τ is in $T[\varphi[\Gamma_0]]$.
For $n = 0$, this is clear.
Let $n > 0$. If τ is a planted Γ-tree with n vertices, then by the induction hypothesis, both $\mathrm{Up}(\tau)$ and $\mathrm{Down}(\tau)$ are in $\mathrm{Im}(T)$, say $\mathrm{Up}(\tau) = T(\beta)$ and $\mathrm{Down}(\tau) = T(\gamma)$. Then

$$T[\varphi[\Gamma_0]] \ni T(B(\beta, \gamma)) = \tau.$$

If $\tau = \mathrm{Combine}(\tau_1, \ldots, \tau_k)$, with all τ_i planted Γ-trees, then by the induction hypothesis, each τ_i is in $T[\varphi_0[\Gamma_0]]$.
Without loss of generality, $\tau_i = T(\alpha_i)$ with $\alpha_n \geq \ldots \geq \alpha_0$ and each $\alpha_i \in \varphi_0[\gamma_0]$. Then $\tau = T(\alpha_n + \ldots + \alpha_0)$.

(2) T is injective.
By induction on $\min(\alpha, \beta)$, and using Lemma 5.14 we immediately get $\alpha \neq \beta \Rightarrow T(\alpha) \neq T(\beta)$.

(3) $(\forall \alpha, \beta < \Gamma_0)(\alpha < \beta \Leftarrow T(\alpha) < T(\beta))$.
By induction on $|T(\alpha)| + |T(\beta)|$.
First suppose that both $T(\alpha)$ and $T(\beta)$ are planted.
Then α, β are in $\varphi_0[\Gamma_0]$ and $T(\alpha) = \mathrm{Pair}(T(\alpha_1), T(\beta_1))$ with $B(\alpha_1, \beta_1) = \alpha$ and $T(\beta) = \mathrm{Pair}(T(\alpha_2), T(\beta_2))$, with $B(\alpha_2, \beta_2) = \beta$.

<u>Case 1</u> $T(\alpha_1) = \mathrm{Up}(T(\alpha)) < \mathrm{Up}(T(\beta)) = T(\alpha_2)$
and $T(\beta_1) = \mathrm{Down}(T(\alpha)) < T(\beta)$.

By the induction hypothesis, we get $\alpha_1 < \alpha_2$ and $\beta_1 < \beta$.
Since β is a limit ordinal, $\beta_1 + 1 < \beta$.
Then $\beta = \varphi_{\alpha_1}(\beta) > \varphi_{\alpha_1}(\beta_1 + 1) \geq B(\alpha_1, \beta_1) = \alpha$.

<u>Case 2</u> $T(\alpha_1) = \mathrm{Up}(T(\alpha)) = \mathrm{Up}(T(\beta)) = T(\alpha_2)$
and $T(\beta_1) = \mathrm{Down}(T(\alpha)) < \mathrm{Down}(T(\beta)) = T(\beta_2)$.

Then $\alpha_1 = \alpha_2$. By the induction hypothesis, we get $\beta_1 < \beta_2$.
Lemma 5.13 gives $B(\alpha_1, \beta_1) < B(\alpha_2, \beta_2)$.

<u>Case 3</u> $T(\alpha_1) = \mathrm{Up}(T(\alpha)) > \mathrm{Up}(T(\beta)) = T(\beta_1)$
and $T(\alpha) \leq \mathrm{Down}(T(\beta)) = T(\beta_2)$.

By the induction hypothesis, we get $\alpha_1 > \beta_1$ and $\alpha \leq \beta_2$.
Then $\alpha \leq \beta_2 < B(\alpha_2, \beta_2) = \beta$ (Lemma 5.15).

Now suppose

$$T(\alpha) = \mathrm{Combine}(T(\alpha_1), \ldots, T(\alpha_k))$$

and

$$T(\beta) = \mathrm{Combine}(T(\beta_1), \ldots, T(\beta_l))$$

with α_i, β_i non-trivial and either k or $l \geq 2$.
Without loss of generality let

$$T(\alpha_1) \geq \ldots \geq T(\alpha_k)$$

and

$$T(\beta_1) \geq \ldots \geq T(\beta_l).$$

Then, by the induction hypothesis, $\alpha_1 \geq \ldots \geq \alpha_k$ and $\beta_1 \geq \ldots \geq \beta_l$.

Case I $T(\alpha_i) = T(\beta_i)\ \forall i \leq k$ and $k < l$.

Then $\alpha = \alpha_1 + \ldots + \alpha_k < \beta_1 + \ldots + \beta_l = \beta$.

Case II $T(\alpha_i) = T(\beta_i)\ \forall i < j \leq k$ and $T(\alpha_j) < T(\beta_j)$.

By the induction hypothesis, $\alpha_j < \beta_j$. Because β_j is additively indecomposable, then $\alpha_j + \ldots + \alpha_k < \beta_j$. Then

$$\alpha = \alpha_1 + \ldots + \alpha_k < \alpha_1 + \ldots + \alpha_{j-1} + \beta_j = \beta_1 + \ldots + \beta_{j-1} + \beta_j \leq \beta.$$

□

Corollary 5.2
There is no infinite strictly decreasing sequence of Γ-trees.

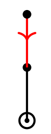
5.2 Phase transition

Definition 5.5
Let $\mathcal{M}$ be a Sim-model.
A finite sequence of Γ-trees $(\tau_0, \ldots, \tau_n)$ in $\mathcal{M}$ is *strictly decreasing* if

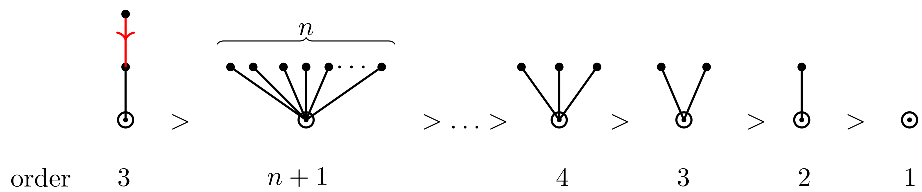
$$\tau_0 > \ldots > \tau_n.$$

We will express the largeness of a particular Γ-tree τ in a Sim model $\mathcal{M}$ by means of the existing strictly decreasing sequences $(\tau_0, \ldots, \tau_k)$, with $\tau_0 = \tau$.

Example 5.1
Already the rather small tree

τ =


allows for arbitrary long strictly decreasing sequences $\tau > \ldots > \tau_n$, such as the following



However, looking at the orders $|\tau_i|$ of the trees in this sequence, we find that a large gap will inevitably occur between the orders of τ and τ_1.

This is the motivation for the following definition.

Definition 5.6
We call the finite sequence of Γ-trees $(\tau_0, \ldots, \tau_n)$ *k-moderate* if

$$(\forall i < n)(|\tau_{i+1}| \leq |\tau_i| + k).$$

Definition 5.7
We call a Γ-tree τ in $\mathcal{M}$ *Γ-large* if for any positive $n < \omega$, there is a finite sequence $(\tau_1, \ldots, \tau_n) \in \Gamma^n_{\text{tree}}$ such that the sequence $(\tau, \tau_1, \ldots, \tau_n)$ is 1-moderate and strictly decreasing.

Just as in the article Weiermann (2003), we will depart from the following result in proof theory.

Theorem 5.2
There exists a Sim-model $\mathcal{M}$ such that

$$\mathcal{M} \models (\exists k < \omega)(\lambda_k \text{ is } \Gamma - \text{large}).$$

Proof
We only sketch the proof and refer to the course *Capita Selecta in de Logica* for more detail.
Consider the function mapping $k < \omega$ to the integer

$$\begin{aligned} \max\{n < \omega :& (\exists (\tau_0, \ldots, \tau_n) \in \Gamma^{n+1}_{\text{tree}}) \\ & (\lambda_k = \tau_0 > \ldots > \tau_n \wedge ((\forall i < n)|\tau_{i+1}| \leq |\tau_i| + 1))\}. \end{aligned}$$

Using the classification of the provably total functions in Sim that was obtained in Friedman et al. (1982) and Simpson (1982), one can deduce by the same method as adopted in Weiermann (1999), that this function is not provably total in Sim.

□

Definition 5.8
Let $\mathcal{M}$ be a Sim-model.
Let $g : \omega \to \mathbb{R}$ be a function in $\mathcal{M}$. The finite sequence of Γ-trees $(\tau_0, \tau_1, \ldots, \tau_k)$ is *bounded* by g if

$$(\forall i \leq k)(|\tau_i| \leq g(i)).$$

For every $n \in \omega$, we denote by $g + n$ the function $g + n : \omega \to \mathbb{R} : k \mapsto n + g(k)$. A positive integer n in $\mathcal{M}$ is *g-large* if there exist strictly decreasing $(g + n)$-bounded finite sequences $(\tau_0, \ldots, \tau_k)$ of Γ-trees of arbitrary (finite) length.

Theorem 5.3 (ZFC)
There is no g-large $n < \omega$, for any $g : \omega \to \mathbb{R}$.

Proof
Suppose $n \in \omega$ were g-large for certain $g : \omega \to \mathbb{R}$. Consider the tree of strictly decreasing finite sequences of Γ-trees $(\tau_0, \tau_1, \ldots, \tau_k)$ ordered by the relation "is

an initial segment of". Because there exist only finitely many Γ-trees with $|\tau_k| \leq g(k) + n$, this tree is finitely branching. Because n is g-large, the tree is infinite. By König's Lemma, τ contains an infinite path. This implies that there is an infinite strictly decreasing set of Γ-trees. This is in contradiction with Corollary 5.2. □

Definition 5.9
If g is an $\mathcal{L}_\in$-defined function, the sentence φ_g is the formulation in $\mathcal{L}_\in$ of the following statement

"There does not exist any g-large $n < \omega$."

Lemma 5.16 (Sim)
Suppose f, g are functions $\omega \to \mathbb{R}$ such that $(\forall k \geq k_0)(f(k) \geq g(k))$. Suppose n is g-large. Then $n + \max_{i<k_0} g(i)$ is f-large.

Proof
This follows immediately from the definitions. □

Corollary 5.3
Suppose f, g are $\mathcal{L}_\in$-defined functions $\omega \to \mathbb{R}$ such that Sim proves that f eventually dominates g. Then

$$\mathsf{Sim} \vdash \varphi_f \Rightarrow \varphi_g.$$

Let's recall the main theorem from the previous chapter.

Theorem 5.4
There exist $q, b \in \mathbb{R}_{>0}$ and two sequences $(b_n)_{n\in\omega}$, $(q_n)_{n\in\omega}$ of positive reals, $(q_n)_{n\in\omega}$ increasing, such that:

- $t(k) \sim b\frac{q^k}{k^{\frac{3}{2}}}$ as $k \to \infty$.
- $t_\leq(k) \sim \frac{qb}{q-1}\frac{q^k}{k^{\frac{3}{2}}}$ as $k \to \infty$.
- $t_n(k) \sim b_n\frac{q_n^k}{k^{\frac{3}{2}}}$ as $k \to \infty$.
- $q_n \to q$ as $n \to \infty$.

Now put

$$c = \frac{\log 2}{\log q}.$$

Using the numerical value for q that is available in the OEIS (see Sloane (2018a)), we find $c = 0.4004216000227538\ldots$.

Definition 5.10
This definition runs in the meta-theory. For any rational $r \in \mathbb{Q}$, we can define a canonical term $\underline{r}$ (e.g. $\underline{r} = \frac{\underline{a}}{\underline{b}}$, where a, b are relatively prime, $\underline{a} = \underline{1} + \ldots + \underline{1}$). We use this term to write down an $\mathcal{L}_\in$-formula defining $f_r : \omega \to \omega : x \mapsto r|x|$, where $|x|$ denotes the length of the binary representation of x. We will abbreviate φ_{f_r} to φ_r.

Lemma 5.17
For any rational $r \in \mathbb{Q}_{>c}$, $\mathsf{Sim} \vdash \underline{r} > c$.
For any rational $r \in \mathbb{Q}_{<c}$, $\mathsf{Sim} \vdash \underline{r} < c$.

Proof
Let $f : \omega \to \mathbb{R}$ be the function $f(k) = \frac{t(k)}{t(k+1)}(1+\frac{1}{k})^{\frac{3}{2}}$. By Theorem 3.9 there is a constant C such that $f(k) - \frac{\rho C}{k} \leq \rho \leq f(k) + \frac{\rho C}{k}$ holds $\forall k < \omega$.
This implies that the function $d(k)$, giving the k-th decimal digit of c, is computable and therefore given by a Σ_1-formula in the arithmetical hierarchy. It follows that this function is absolute between models of Sim. □

Lemma 5.18 (Sim)
For any real $r > c$, there exist $h, d < \omega$ and an infinite sequence of Γ-trees $(\upsilon_i)_{d \leq i < \omega}$ such that:

$$\begin{aligned}(\forall i < \omega) d \leq i \Rightarrow\ & \upsilon_i < \lambda_h \\ & |\upsilon_i| \leq r|i| \\ & (i < j) \wedge (|i| = |j|) \Rightarrow \upsilon_i > \upsilon_j.\end{aligned}$$

Proof
We work in a Sim-model $\mathcal{M}$. Write

$$S_i^h = \{\tau \in \Gamma_{\text{tree}} : \tau < \lambda_h \wedge |\tau| = i\}.$$

First choose $h < \omega$ large enough such that

$$r > \frac{\log 2}{\log(q_h)}.$$

Note that

$$\lim_{i \to +\infty} t_h(\lfloor r|i| \rfloor).2^{-|i|} = \lim_{i \to +\infty} b_h \frac{2^{(\frac{r \log(q_h)}{\log 2} - 1)|i| + O(1)}}{\lfloor r|i| \rfloor^{\frac{3}{2}}} = +\infty.$$

This means that we can choose $d < \omega$ such that

$$|S^h_{\lfloor r|i| \rfloor}| = t_h(\lfloor r|i| \rfloor) \geq 2^{|i|} \qquad \forall i \geq d.$$

Then, we can choose Γ-trees $\upsilon_d > \ldots > \upsilon_{2^{|d|}-1}$ in $S^h_{\lfloor r|d| \rfloor}$ and successively Γ-trees $\upsilon_{2^{|i|-1}} > \upsilon_{2^{|i|-1}+1} > \ldots > \upsilon_{2^{|i|}-1}$ in $S^h_{\lfloor r|i| \rfloor}$ for all $i > d$.
Then the sequence $(\upsilon_i)_{d \leq i < \omega}$ has the desired properties.

□

Lemma 5.19 (Sim)
Let r be a real number, $r > c$ and $f : \omega \to \mathbb{R}$, $g : \omega \to \mathbb{R}$ such that

$$g(i) = r|i| + f(|i|).$$

Then $\varphi_g \Rightarrow \varphi_f$.

Proof
Let $n < \omega$ be f-large.
Choose numbers $d, h < \omega$ and an infinite sequence $(\upsilon_i)_{d \le i < \omega}$ such as in the previous lemma.
Let m be arbitrary. Since n is f-large, there exists a sequence $\sigma_0 > \ldots > \sigma_m$ that is $(f+n)$-bounded.
Now consider $(\tau_i)_{i<m}$ with $\tau_i = \begin{cases} \text{Combine}(\text{Pair}(\lambda_p, \lambda_p), d-i) & \text{for } i < d \\ \text{Combine}(\text{Pair}(\lambda_p, \sigma_{|i|}), \upsilon_i) & \forall i \ge d, \end{cases}$
where $p < \omega$ is sufficiently large (not depending on i or m).
Then $(\tau_i)_{i<m}$ is strictly decreasing and $g + n'$-bounded (n' not depending on m or i).
It follows that n' is g-large. □

Lemma 5.20
Let $f : \omega \to \omega : i \mapsto i$ then

$$\mathsf{Sim} \nvdash \varphi_f.$$

Proof
Suppose n is f-large. Then λ_{n+1} is Γ-large. Indeed, for any $m < \omega$, we find $(\tau_0, \tau_1, \ldots, \tau_m)$ strictly decreasing with $|\tau_i| \le f(i) + n$. Then $(\lambda_{n+1}, \tau_0, \ldots, \tau_m)$ is strictly decreasing and 1-moderate.
Hence, the provability of φ_f would contradict Theorem 5.2. □

Lemma 5.21
For $r \in \mathbb{Q}_{<c}$,

$$\mathsf{Sim} \vdash \varphi_r.$$

Proof
We work in a Sim-model $\mathcal{M}$. Let $k \in \omega$. Using Lemma 5.17, it suffices to prove

$$\lim_{m\to\infty} \frac{t_{\le}(\lfloor k + c|m| \rfloor)}{m} = 0,$$

this becomes evident after taking a logarithm:

$$\begin{aligned}
&\lim_{m\to\infty} \log \frac{t_{\le}(\lfloor k + c|m| \rfloor)}{m} \\
&= \lim_{m\to\infty} \left(\log\left(\frac{b}{1-q^{-1}}\right) + \left(k + \frac{|m|\log 2}{\log q}\right)\log q - \frac{3}{2}\log\left(k + \frac{|m|\log 2}{\log q}\right) - \log m \right) \\
&= \lim_{m\to\infty} \left(\log\left(\frac{b}{1-q^{-1}}\right) + k\log q + |m|\log 2 - \frac{3}{2}\log\left(k + \frac{|m|\log 2}{\log q}\right) - \log m \right) \\
&= \lim_{m\to\infty} -\frac{3}{2}\log\left(k + \frac{|m|\log 2}{\log q}\right) + O(1) \\
&= -\infty
\end{aligned}$$

□

Theorem 5.5
For rational numbers r the following dichotomy holds:

1. if $r < c$ then $\mathsf{Sim} \vdash \varphi_r$.
2. if $r > c$ then φ_r is independent from Sim.

Proof
The first point was proved in Lemma 5.21. We now prove the second point. Let $r \in \mathbb{Q}_{>c}$. Combining Lemma 5.19 and Lemma 5.20, we find that

$$T \nvdash \varphi_{C|i|},$$

for a certain constant C. Applying Lemma 5.19 once again, now with $f(i) = C|i|$ and r' a rational number in $]c, r[$, we find

$$T \nvdash \varphi_{r'|i|+|C|i||}.$$

Since $r'|i| + |C|i||$ is eventually dominated by $r|i|$, Lemma 5.3 shows that $\mathsf{Sim} \nvdash \varphi_r$. Since $\mathsf{ZFC} \vdash \varphi_r$ (by Theorem 5.3), we find that φ_r must in this case be independent from Sim.

□

References and Remarks

The results and proofs in Section 5.1 can be found in Schütte (1977) (for a related ordinal notation system). The idea for the proof of Theorem 5.5 is due to Andreas Weiermann.

6

End extensions of models of arithmetic and set theory

6.1 End extensions

In chapters 6, 7 and 8 we will focus on some more structural properties of models of arithmetic and set theory. In particular, we will study elementary extensions of such models. We will find that also in this respect, there are interesting parallels to be drawn between models of arithmetic and set theory. Nevertheless, we will also encounter important differences regarding the properties of elementary extensions in both families of models. The set theories that we will study here will range from ZFC to GBC and beyond. We assume the reader is familiar with the theories PA and ZFC and their corresponding languages $\mathcal{L}_A$ and $\mathcal{L}_\in$. We start by recalling the concepts of elementary embedding and elementary extension.

Definition 6.1
A map $f : M \to N$ between two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ is an *elementary embedding* if for all $\mathcal{L}$-formulas $\varphi(v_1, \dots, v_n)$ and all $m_1, \dots, m_n \in M$,

$$\mathcal{M} \models \varphi(m_1, \dots, m_n)$$

holds precisely when

$$\mathcal{M} \models \varphi(f(m_1), \dots, f(m_n))$$

holds.

The word elementary *embedding* suggests adding to the above definition that f should be injective, but a moment's thought learns that injectivity already follows from the above definition[1] and this is therefore not necessary.

[1] Any first-order language $\mathcal{L}$ contains the symbol "=".

Definition 6.2
Let $\mathcal{M}$ be a substructure of the $\mathcal{L}$-structure $\mathcal{N}$. We define:

$\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$
if and only if $\mathcal{N}$ is an *elementary extension* of $\mathcal{M}$
if and only if the inclusion map $M \hookrightarrow N$ is an elementary embedding.

We will denote this as $\mathcal{M} \prec \mathcal{N}$.

While every structure is trivially an elementary extension of itself, we will consequently use the word "elementary extension" with the intended meaning of "proper elementary extension".
Both PA- and ZFC-models allow for an "orientation".

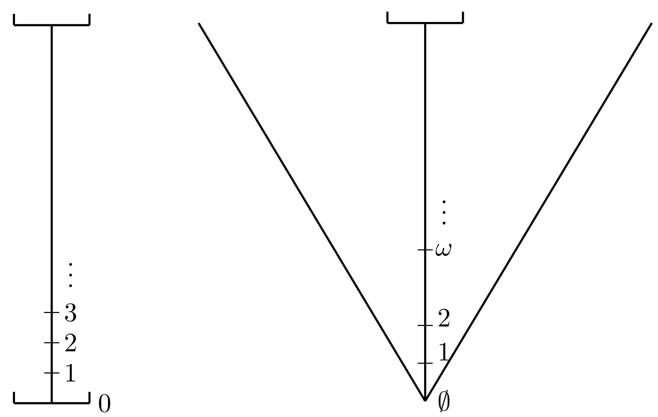
1. In PA-models $\mathcal{M}$, this orientation is given by the definable order $<$ on $\mathcal{M}$.
2. Important information on a ZFC-model $\mathcal{M}$, is carried by the linear order $(\mathrm{On}^{\mathcal{M}}, \in^{\mathcal{M}})$.

It is important to note that neither of these two linear orders

- $(M, <^{\mathcal{M}})$, where $\mathcal{M} \models \mathsf{PA}$ and
- $(\mathrm{On}^{\mathcal{M}}, \in^{\mathcal{M}})$ where $\mathcal{M} \models \mathsf{ZFC}$,

is necessarily a well-order.
We have in mind the following mental image of ...

... a PA-model ... a ZFC-model



The "orientations" give rise to the notion of end extensions.
Intuitively speaking, an end extension of $\mathcal{M}$ is an extension that is built "on top" of $\mathcal{M}$.
First, we introduce this notion for the theory PA.

Definition 6.3
Let $\mathcal{N}$ be an $\mathcal{L}_A$-structure and $\mathcal{M}$ a sub-structure of $\mathcal{N}$ satisfying PA.
$\mathcal{N}$ is an *end extension* of $\mathcal{M}$ if

$$(\forall n, m \in N) \quad (\mathcal{N} \models n < m) \wedge (m \in M) \quad \Rightarrow n \in M.$$

Which PA-models admit an elementary end extension?

Perhaps surprising, the answer is that *every* PA-model has such an elementary end extension. This result is known as the MacDowell-Specker Theorem, after R. MacDowell and E. Specker, who first attained this theorem in 1961 in the context of a study (MacDowell and Specker (1961)) of the group structures that can arise from models of PA[2].

Before turning to the MacDowell-Specker Theorem in the next chapter, we discuss a weakened result, namely the special case of countable PA-models.

Theorem 6.1
Every countable $\mathcal{M} \models \mathsf{PA}$ has an elementary end extension.

Our reason for first considering this weaker version is that this countable case can be deduced in a relatively direct way using the Omitting Type Theorem. The same line of argument is not feasible in the general case. This has additional value for our story since we will see that the omitting-types-line of argument can be modified for ZFC-models. In order to prove Theorem 6.1, we introduce the model theory of so-called "types".

6.2 Types

We will construct elementary extensions of models $\mathcal{M}$ by placing an ideal element on top of the model. To keep control of the extension thus fabricated, we need to track the sets of sentences that can be realised in this extension. Types are the tools that allow us to accomplish this.

Definition 6.4
An n-type $p(v_1, \ldots, v_n)$ is a collection of formulas

$$p(v_1, \ldots, v_n) = \{\varphi_i(v_1, \ldots, v_n) : i \in I\}$$

in the variables $v_1, \ldots, v_n$ in a given language $\mathcal{L}$.
An n-type $p(v_1, \ldots, v_n)$ is complete if for every $\mathcal{L}$-formula φ, with free variables contained in $v_1, \ldots, v_n$, either $\varphi \in p$ or $\neg\varphi \in p$.
If $\mathcal{M}$ is an $\mathcal{L}$-structure and $\overline{m} \in M^n$, then $\overline{m}$ realises the n-type $p(\overline{v})$ in $\mathcal{M}$ if $\mathcal{M} \models \varphi(\overline{m})$ for every formula $\varphi(\overline{v})$ in $p(\overline{v})$.
We say that an n-type $p(\overline{v})$ is realised in $\mathcal{M}$ if there is $\overline{m} \in M^n$ realising $p(\overline{v})$, otherwise we say that $\mathcal{M}$ omits $p(\overline{v})$.

[2]For $\mathcal{M} \models \mathsf{PA}$, $(M, +^{\mathcal{M}})$ is a cancellative semigroup and can therefore be embedded in a group in a canonical way.

If T is an $\mathcal{L}$-theory and $p(v_1,\ldots,v_n)$ is an n-type, then we say that T is consistent with $p(v_1,\ldots,v_n)$, precisely when the theory

$$T \cup p(v_1,\ldots,v_n) = T \cup \{\varphi(c_1,\ldots,c_n) : \varphi(\overline{v}) \in p(\overline{v})\}$$

is satisfiable, where $c_1,\ldots,c_n$ are newly introduced constants.

It is generally common practice to interpret the variables $v_1,\ldots,v_n$ in a type $p(v_1,\ldots,v_n)$ as constants rather than variables.

Definition 6.5
Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $p(\overline{v})$ an n-type in $\mathcal{L}$.
Then $p(\overline{v})$ is an n-type of $\mathcal{M}$ if $\mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$ (the elementary diagram of $\mathcal{M}$) is consistent with $p(\overline{v})$.
For a theory T, $S_n(T)$ is the set of complete n-types $p(\overline{v})$ that are consistent with T.
For an $\mathcal{L}$-structure $\mathcal{M}$, $S_n(\mathcal{M})$ is the set of complete n-types $p(\overline{v})$ of $\mathcal{M}$, or equivalently, $S_n(\mathcal{M}) = S_n(\mathrm{Diag}_{\mathrm{el}}(\mathcal{M}))$.

The Omitting Types Theorem gives a general sufficient condition for the existence of structures that omit certain types (hence the name).
In order to formulate this theorem, we need one more notion, that of isolated types.

Definition 6.6
Let T be an $\mathcal{L}$-theory, let $p(v_1,\ldots,v_n)$ be an n-type and $\varphi(v_1,\ldots,v_n)$ an $\mathcal{L}$-formula.
Then $\varphi(v_1,\ldots,v_n)$ isolates $p(v_1,\ldots,v_n)$ (with respect to T) if

- $T \cup \{\varphi(v_1,\ldots,v_n)\}$ is satisfiable,
- for every formula $\psi(v_1,\ldots,v_n)$ in $p(v_1,\ldots,v_n)$, we have that

$$T \vdash (\forall v_1,\ldots,v_n)(\varphi(v_1,\ldots,v_n) \Rightarrow \psi(v_1,\ldots,v_n)).$$

A type $p(v_1,\ldots,v_n)$ is *isolated* (with respect to T) if there is an $\mathcal{L}$-formula $\varphi(v_1,\ldots,v_n)$ isolating $p(v_1,\ldots,v_n)$ and $p(v_1,\ldots,v_n)$ is *non-isolated* otherwise.

The following is now really just unwinding of the definition.

Lemma 6.1
Let T be an $\mathcal{L}$-theory and $p(v_1,\ldots,v_n)$ an n-type isolated by $\varphi(v_1,\ldots,v_n)$.

1. If $p(\overline{v})$ is a complete type, then for any $\mathcal{L}$-formula $\psi(v_1,\ldots,v_n)$, $\psi(\overline{v})$ is in $p(\overline{v})$ if and only if

$$T \vdash (\forall \overline{v})(\varphi(\overline{v}) \Rightarrow \psi(\overline{v})).$$

2. Any model $\mathcal{M}$ of T with the property

$$\mathcal{M} \models (\exists v_1,\ldots,v_n)(\varphi(v_1,\ldots,v_n))$$

realises $p(\overline{v})$.

3. If T is complete, then any model $\mathcal{M}$ of T realises $p(\overline{v})$.

Proof

1. For any $\mathcal{L}$-formula $\psi(v_1, \ldots, v_n) \in p(\overline{v})$, we have

$$T \vdash (\forall \overline{v})(\varphi(\overline{v}) \Rightarrow \psi(\overline{v})),$$

since $p(v_1, \ldots, v_n)$ is isolated by $\varphi(v_1, \ldots, v_n)$. If $\psi(v_1, \ldots, v_n) \notin p(\overline{v})$, then $\neg\psi(v_1, \ldots, v_n) \in p(\overline{v})$, so

$$T \vdash (\forall \overline{v})(\varphi(\overline{v}) \Rightarrow \neg\psi(\overline{v})),$$

and therefore it can not be the case that

$$T \vdash (\forall \overline{v})(\varphi(\overline{v}) \Rightarrow \psi(\overline{v})),$$

for otherwise

$$T \vdash (\forall \overline{v})(\varphi(\overline{v}) \Rightarrow \bot),$$

and this would contradict the condition that an isolating formula is consistent with T.

2. This follows straight from the definition of $p(v_1, \ldots, v_n)$ being isolated by $\varphi(v_1, \ldots, v_n)$.

3. Since T is complete and consistent with $\psi(v_1, \ldots, v_n)$,

$$T \vdash (\exists v_1, \ldots, v_n)(\varphi(v_1, \ldots, v_n)),$$

whence the statement follows using the previous point.

□

We motivate now our interest in types and particularly in omitting them, by relating back to our eventual goal of finding for any given PA-model $\mathcal{M}$ an elementary end extension.
Let $\mathcal{M} \models \mathsf{PA}$ be given. The elementary extensions $\mathcal{N}$ of $\mathcal{M}$ are (up to isomorphism) exactly the models of the $\mathcal{L}_{\mathcal{M}}$-theory $\mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$.
How can we express that such an $\mathcal{L}_{\mathcal{M}}$-model $\mathcal{N}$ with

$$\mathcal{N} \models \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$$

is also an end extension?
Here the types-terminology comes in handily as the elementary end extensions $\mathcal{N}$ of $\mathcal{M}$ are exactly the models $\mathcal{N}$ of $\mathrm{Diag}_{\mathrm{el}}(\mathcal{N}) \cup \{c > m : m \in M\}$ that omit each of the 1-types

$$p_a(v) = \{v < a\} \cup \{v \neq m : m \in M\},$$

where $a \in M$.
Let T be an $\mathcal{L}$-theory, $p(\overline{v})$ an n-type of T. Under what conditions can we expect T to have a model omitting $p(\overline{v})$?
By the third point of Lemma 6.1, if T is complete, it is necessary for this that the type $p(\overline{v})$ is non-isolated. Interestingly, this is in fact everything that is needed, as long as $\mathcal{L}$ is countable.

Theorem 6.2 (Omitting Types Theorem)
Let T be a theory in a countable language $\mathcal{L}$.
Then for any non-isolated n-type $p(\overline{v})$, there exists a countable $\mathcal{L}$-structure $\mathcal{M}$ satisfying T and omitting $p(\overline{v})$.

We can even strengthen this theorem to omit countably many types in one model.

Theorem 6.3 (Omitting countably many types at once)
Let T be a theory in a countable language $\mathcal{L}$.
Then for any countable set S consisting of non-isolated types, there exists a countable $\mathcal{L}$-structure $\mathcal{M}$ satisfying T but omitting all types in S.

Before proving this theorem, we show how to use it to prove Theorem 6.1.

Proof of Theorem 6.1
Let $\mathcal{M}$ be a countable PA-model. Let $\mathcal{L}_{\mathcal{M}}$ be the language obtained from $\mathcal{L}$ by adding the constant symbol m for any $m \in M$. It is a crucial observation that $\mathcal{L}_{\mathcal{M}}$ is still countable.
Let T be the $\mathcal{L}_{\mathcal{M}} \cup \{c\}$-theory $T = \mathrm{Diag}_{\mathrm{el}}(\mathcal{M}) \cup \{c > m : m \in M\}$ and define for each $a \in M$ the 1-type

$$p_a(v) = \{v < a\} \cup \{v \neq m : m \in M\}.$$

It suffices to prove that there is a model $\mathcal{M}$ of T omitting all types in the countable set $\{p_a(v) : a \in M\}$.
By Theorem 6.3, it suffices to show that for each $a \in M$, the type $p_a(v)$ is non-isolated.
Suppose, in desire of contradiction, that $p_a(v)$ is isolated by the $\mathcal{L}_{\mathcal{M}} \cup \{c\}$-formula $\varphi(v)$.
We can "factor" $\varphi(v)$ as $\varphi(v) = \chi(v, c)$, where χ is an $\mathcal{L}_{\mathcal{M}}$-formula.
We first prove the following claim:

$$\mathcal{M} \models (\forall x_1)(\exists x_2)(\forall y)(\chi(x_1, y) \Rightarrow y \leq x_2).$$

Proof of the claim:
Let $m_1 \in M$ arbitrary, we look for $m_2 \in M$ such that

$$\mathcal{M} \models (\forall y)(\chi(m_1, y) \Rightarrow y \leq m_2).$$

Since $\varphi(v)$ isolates $p_a(v)$, we have in particular that

$$T \vdash (\forall v)(\varphi(v) \Rightarrow v \neq m_1).$$

It follows that $T \vdash \neg\varphi(m_1)$, which is the same as $T \vdash \neg\chi(m_1, c)$.
However, from $T \vdash \neg\chi(m_1, c)$, we deduce by compactness that there is $m_2 \in M$ such that $\mathrm{Diag}_{\mathrm{el}}(\mathcal{M}) \cup \{c > m_2\} \vdash \neg\chi(m_1, c)$, which we easily rewrite to

$$\mathrm{Diag}_{\mathrm{el}}(\mathcal{M}) \vdash \chi(m_1, c) \Rightarrow c \leq m_2$$

and since c was a constant symbol alien to $\mathcal{M}$,

$$\mathrm{Diag}_{\mathrm{el}}(\mathcal{M}) \vdash (\forall y)(\chi(m_1, y) \Rightarrow y \leq m_2).$$

This proves the claim.
Again since $\varphi(v)$ isolates $p_a(v)$, we also have:

(1) $T \vdash (\forall v)(\varphi(v) \Rightarrow v < a)$.

(2) $T \cup \{\varphi(v)\}$ is satisfiable.

By (2), $\chi(m_v, c)$ is true in an elementary extension of $\mathcal{M}$ satisfying T (for a well-chosen element m_v).
For any $m \in M$, we have by (1) and $c > m \in T$, that the sentence

$$(\exists y > m)(\exists v < a)\ \chi(v, y)$$

is true in this elementary extension of $\mathcal{M}$, and by consequence also in $\mathcal{M}$. Thus,

$$\mathcal{M} \models (\forall x)(\exists y > x)(\exists v < a)\ \chi(v, y).$$

Using the Pigeonhole Principle, which is provable in PA, we find that there is $m_1 \in M$ such that

$$\mathcal{M} \models (\forall x)(\exists y > x)\ \chi(m_1, y),$$

but our claim learns

$$\mathcal{M} \models (\exists x_2)(\forall y)\ (\chi(m_1, y) \Rightarrow y \leq x_2.)$$

Both statements are clearly contradictory.
This contradiction proves that the types $p_a(v)$ with $a \in M$ are in fact non-isolated and hence concludes the proof. $\square$

It remains to give a proof of Theorem 6.3 and we turn to this task now.

Proof of Theorem 6.3
This is one of the model-existence theorems that are susceptible to a Henkin-argument (the compactness and completeness theorems are other examples).
We recall that an $\mathcal{L}$-theory T has the Witness Property if for any existential formula $\exists v\varphi(v)$, there is a constant c in $\mathcal{L}$ such that $\exists v\varphi(v) \Rightarrow \varphi(c)$ is provable in T.
We will use Henkin's main lemma (see the proof of the Compactness Theorem in Marker (2002) or the course *Logica I* (van Oosten and Moerdijk (2011))).

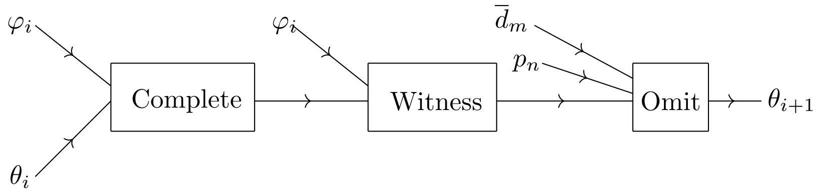
Lemma 6.2
Every complete (finitely) satisfiable $\mathcal{L}$-theory T with the Witness Property has a model $\mathcal{M} \models T$ with the following property.
For every element $m \in M$, there is a constant symbol $c \in T$ such that $m = c^{\mathcal{M}}$.
Moreover, one can secure $|M| \leq \kappa$, where κ is the cardinality of the set of constant symbols of $\mathcal{L}$.

Let $C = \{c_n : n < \omega\}$ be a set of mutually different constant symbols alien to $\mathcal{L}$ and let $\mathcal{L}^* = \mathcal{L} \cup C$.
We use $c, c_1, \ldots, c_n, d, d_1, \ldots, d_n$ as syntactic variables to range over the constant symbols that are contained in C. We will expand T to a complete $\mathcal{L}^*$-theory T^* with the Witness Property. This will be accomplished by putting $T^* = T \cup \{\theta_i : i < \omega\}$ for well-chosen formulas θ_i.
We construct the sequence $(\theta_i)_{i<\omega}$ recursively.
Let $(p_i)_{i<\omega}$ and $(\overline{d}_i)_{i<\omega}$ be enumerations of the elements of respectively the set S and the set $C^{<\omega}$.

Let $(\varphi_i)_{i<\omega}$ be a sequence with terms in $\{\varphi : \varphi \text{ is an } \mathcal{L}^*\text{-sentence}\}$, where each element of $\{\varphi : \varphi \text{ is an } \mathcal{L}^*\text{-sentence}\}$ occurs infinitely many times. We fix a bijection $b : \omega^2 \to \omega$. We start by putting $\theta_0 = (\forall v)(v = v)$.
For each $i < \omega$, we define

$$\theta_{i+1} = \text{Omit}(\overline{d}_m, p_n, \text{Witness}(\varphi_i, \text{Complete}(\varphi_i, \theta_i))), \tag{6.1}$$

where $b(m, n) = i$. Schematically:



We define each of these operations separately.

Complete(φ_i, θ_i)

If $T \cup \{\theta_i \wedge \varphi_i\}$ is satisfiable, return $\theta_i \wedge \varphi_i$.
Else, return $\theta_i \wedge \neg\varphi_i$.

- Note that Complete(φ_i, θ_i) is well-defined and $T \cup \{\text{Complete}(\varphi_i, \theta_i)\}$ is satisfiable, as long as $T \cup \{\theta_i\}$ is satisfiable.

Witness(φ_i, θ_i)

If there is an $\mathcal{L}^*$-formula $\psi(v)$ such that $\varphi_i \equiv \exists v\psi(v)$ and $T \vdash \theta_i \Rightarrow \exists v\psi(v)$, return $\theta_i \wedge \psi(c)$, where c is a constant of C not occurring in $T \cup \{\theta_i\}$.
Else return θ_i.

- Note that Witness(φ_i, θ_i) is always well-defined since the constant symbols of C do not occur in T and there can always be found a constant in C not occurring in the formula θ_i.

- Note that if $T \cup \{\theta_i\}$ is satisfiable, then so is $T \cup \{\text{Witness}(\varphi_i, \theta_i)\}$.
This is clear if $\theta_i = \text{Witness}(\varphi_i, \theta_i)$.
Else, $\varphi_i \equiv \exists v\psi(v)$ and $T \vdash \theta_i \Rightarrow \exists v\psi(v)$, for certain $\mathcal{L}^*$-formula $\psi(v)$.
If $T \cup \{\theta_i\}$ is satisfiable, then it has a model $\mathcal{N}$ with then necessarily $\mathcal{N} \models \exists v\psi(v)$, i.e., there is $x \in N$ such that $\mathcal{N} \models \psi(x)$. By setting $c^{\mathcal{N}} = x$, we get a model of $T \cup \{\text{Witness}(\varphi_i, \theta_i)\}$.

Omit$(\overline{d}_m, p_n, \theta_i)$

Write $\overline{d}_m = (d_1, \ldots, d_l)$.
If p_n is a k-type and $k \neq l$, then return θ_i.
Else, choose an $\mathcal{L}$-formula $\psi(v_1, \ldots, v_{l+r})$ such that $\theta_i \equiv \psi(d_1, \ldots, d_l, c_1, \ldots, c_r)$, where the constant symbols $c_1, \ldots, c_r$ are in $C \setminus \{d_1, \ldots, d_l\}$.
Define $\chi(v_1, \ldots, v_l) = (\exists w_{c_1}, \ldots, w_{c_r})\psi(v_1, \ldots, v_l, w_{c_1}, \ldots, w_{c_r})$, where $w_{c_1}, \ldots, w_{c_r}$ are new variables.
Choose an $\mathcal{L}$-formula $\varphi(v_1, \ldots, v_l)$ in p_n such that

$$T \not\vdash (\forall v_1, \ldots, v_l)\chi(v_1, \ldots, v_l) \Rightarrow \varphi(v_1, \ldots, v_l).$$

Return $\theta_i \wedge \neg\varphi(d_1, \ldots, d_l)$.

- If $T \cup \{\theta_i\}$ is satisfiable, then so is $T \cup \{\mathrm{Omit}(\overline{d}_m, p_n, \theta_i)\}$.
  This is clear if $\theta_i = \mathrm{Omit}(\overline{d}_m, p_n, \theta_i)$.
  Else, by construction
  $$T \nvdash (\forall v_1, \dots, v_l)\ \chi(v_1, \dots, v_l) \Rightarrow \varphi(v_1, \dots, v_l),$$
  so there is a model $\mathcal{N}$ of T, with $x_1, \dots, x_l \in N$ such that
  $$\mathcal{N} \models \chi(x_1, \dots, x_l) \wedge \neg\varphi(x_1, \dots, x_l).$$
  But then there are also $y_1, \dots, y_r \in N$ such that
  $$\mathcal{N} \models \psi(x_1, \dots, x_l, y_1, \dots, y_r) \wedge \neg\varphi(x_1, \dots, x_l).$$
  By putting $d_1^{\mathcal{N}} = x_1, \dots, d_l^{\mathcal{N}} = x_l$ and $c_1^{\mathcal{N}} = y_1, \dots, c_r^{\mathcal{N}} = y_r$, we have constructed a model of $T \cup \{\theta_i\}$.
- $\mathrm{Omit}(\overline{d}_m, p_n, \theta_i)$ is well-defined as long as $T \cup \{\theta_i\}$ is satisfiable and p_n is non-isolated. Indeed, this secures that the wanted formula $\varphi(v_1, \dots, v_l)$ in p_n can be found, since then $\{\chi(v_1, \dots, v_l)\} \cup T$ is satisfiable and the existence of φ then follows from the definition of a non-isolated type.

Let $(\theta_i)_{i<\omega}$ be constructed by the rule (6.1).
It follows from our notes that each of these θ_i is well-defined, since we can induct on the fact that $T \cup \{\theta_i\}$ is satisfiable.
By checking each of the return lines it is immediately clear that $\theta_i \vdash \theta_j$ for each $i > j$. Since each $T \cup \{\theta_i\}$ is satisfiable, we find by compactness that $T^* = T \cup \{\theta_i : i < \omega\}$ is satisfiable.
The completeness operator ensures that for each $\mathcal{L}^*$-formula φ_i, either
$$\mathrm{Complete}(\varphi_i, \theta_i) \vdash \varphi_i$$
or
$$\mathrm{Complete}(\varphi_i, \theta_i) \vdash \neg\varphi_i,$$
since $\theta_{i+1} \vdash \mathrm{Complete}(\varphi_i, \theta_i)$, T^* is complete.
T^* has the Witness Property. Indeed, for any existential $\mathcal{L}^*$-formula $\exists v \psi(v)$ with the property $T^* \vdash \exists v \psi(v)$, there is $i < j < \omega$ such that $\exists v \psi(v) \equiv \varphi_j$ and $T \cup \{\theta_i\} \vdash \varphi_j$. But then $\mathrm{Witness}(\varphi_j, \theta_j) \vdash \psi(d)$ for $d \in C$.
It follows that we can apply Henkin's main lemma to find a countable model $\mathcal{M}$ of T such that for every element $m \in M$, there is a constant symbol c in $\mathcal{L}^*$ such that $m = c^{\mathcal{M}}$.
We claim that $\mathcal{M}$ omits all types p_n in S.
Indeed, suppose $\overline{v} = (v_1, \dots, v_l) \in M^l$, realises p_n in $\mathcal{M}$. Then there is $m < \omega$ such that $\overline{v} = \overline{d}_m^{\mathcal{M}}$ and $i < \omega$ such that $b(m, n) = i$.
But then there is a formula $\varphi(v_1, \dots, v_l)$ in p_n such that
$$T^* \vdash \mathrm{Omit}(\overline{d}_m, p_n, \theta_i) \vdash \neg\varphi(d_1, \dots, d_l).$$
This is a contradiction. □

6.3 End- and Rank extensions of ZF-models

We turn to ZF-models next and analyse to what extent the previous considerations can be modified for these models. Let $\mathcal{M} \models \mathsf{ZF}$.
There are (at least) two interesting ways of defining when an extensions of such $\mathcal{M}$ is built "on top of" $\mathcal{M}$.

The first way is to slightly modify the definition of end extension of PA-models, by simply replacing "$<$" by "$\in$" where necessary. This results in the following.

Definition 6.7
Let $\mathcal{N}$ be an $\mathcal{L}_\in$-structure and $\mathcal{M}$ a sub-structure of $\mathcal{N}$.
$\mathcal{N}$ is an *end extension* of $\mathcal{M}$ if
$(\forall n, m \in N) \qquad (\mathcal{N} \models n \in m) \wedge (m \in M) \;\Rightarrow n \in M.$

Hence $\mathcal{N}$ is an end extension of $\mathcal{M}$ if it does not induce any new elements in a set of M.

Example 6.1
Let $\mathcal{M} = (M, \in^{\mathcal{M}} = \in\restriction M)$ and $\mathcal{N} = (N, \in^{\mathcal{N}} = \in\restriction N)$ with $M \subseteq N$ both transitive, then $\mathcal{N}$ is an end extension of $\mathcal{M}$.

In the context of set theory, there is a second natural way to express that an extension $\mathcal{N}$ of $\mathcal{M}$ is "above $\mathcal{M}$", by asking that the elements of $N \setminus M$ have large rank, this gives rise to the notion of rank extension.

Definition 6.8
Let $\mathcal{N}$ be an $\mathcal{L}_\in$-structure satisfying ZF and $\mathcal{M}$ a sub-structure of $\mathcal{N}$.
$\mathcal{N}$ is a *rank extension* of $\mathcal{M}$ if

$$(\forall x \in N \setminus M) \quad (\forall y \in M)(\mathcal{N} \models \mathrm{Rank}(y) < \mathrm{Rank}(x)).$$

Since $\mathcal{N} \models (\forall x, y)(x \in y \Rightarrow \mathrm{Rank}\, x < \mathrm{Rank}\, y)$, any *rank extension* is automatically an end extension. Yet, the converse is not true. This becomes clear from the following example.

Example 6.2
Let $\mathcal{M} = (M, \in^{\mathcal{M}} = \in\restriction M)$ be a countable ZFC-model, with M a transitive set and let G be an M-generic filter on a forcing P (see course notes *Logica II* (Koepke (2016b))), then the structure $M[G]$ end-extends $\mathcal{M}$. However, since $M[G] \cap \mathrm{On} = M \cap \mathrm{On}$, $M[G]$ can not be a rank extension of $\mathcal{M}$.

Conveniently, both notions do coincide when it comes to *elementary* extensions of ZF.

Lemma 6.3
If $\mathcal{N}$ is an elementary end extension of $\mathcal{M} \models \mathsf{ZF}$, then $\mathcal{N}$ is a rank extension of $\mathcal{M}$.

Proof
Fix $\alpha \in \mathrm{On}^{\mathcal{M}}$. Let $a \in M$ such that $a = V_\alpha^{\mathcal{M}}$. By elementarity for the $\mathcal{L}_{\mathcal{M}}$-sentence $a = V_\alpha$, we find $a = V_\alpha^{\mathcal{N}}$. We infer from this that for any $n \in N \setminus M$,

$$\mathcal{N} \models \alpha \leq \mathrm{Rank}\, n,$$

for otherwise $\mathcal{N}$ would have added and element to the set $a \in M$, which is not possible since $\mathcal{N}$ end-extends $\mathcal{M}$. □
We can now prove an analogue to Theorem 6.1.

Theorem 6.4
Every countable $\mathcal{M} \models \mathsf{ZF}$ has an elementary rank extension.

Proof
We adapt the omitting-types-proof of the corresponding theorem for PA-models.
Let T be the $\mathcal{L}_{\mathcal{M}} \cup \{c\}$-theory $T = \mathrm{Diag}_{\mathrm{el}}(\mathcal{M}) \cup \{c \notin m : m \in M\}$ and define for each $a \in M$ the 1-type $p_a(v) = \{v \in a\} \cup \{v \neq m : m \in M\}$.
It suffices to prove that there is a model $\mathcal{M}$ of T omitting all types in the countable set $\{p_a(v) : a \in M\}$.
By Theorem 6.3, it suffices to show that for each $a \in M$, the type $p_a(v)$ is non-isolated.
Suppose, in desire of contradiction, that $p_a(v)$ is isolated by the $\mathcal{L}_{\mathcal{M}} \cup \{c\}$-formula $\varphi(v)$.
We can "factor" $\varphi(v)$ as $\varphi(v) = \chi(v, c)$, where χ is an $\mathcal{L}_{\mathcal{M}}$-formula.
We first prove the following claim:

$$\mathcal{M} \models (\forall x_1)(\exists x_2)(\forall y)\ \chi(x_1, y) \Rightarrow y \in x_2.$$

Proof of the claim:
Let $m_1 \in M$ arbitrary, we look for $m_2 \in M$ such that

$$\mathcal{M} \models (\forall y)\ \chi(m_1, y) \Rightarrow y \in m_2.$$

Since $\varphi(v)$ isolates $p_a(v)$, we have in particular that

$$T \vdash (\forall v)(\varphi(v) \Rightarrow v \neq m_1).$$

It follows that $T \vdash \neg\varphi(m_1)$, which is the same as $T \vdash \neg\chi(m_1, c)$.

However, from $T \vdash \neg\chi(m_1, c)$, we deduce by compactness that there is a finite subset $S \subseteq M$ such that

$$\mathrm{Diag}_{\mathrm{el}}(\mathcal{M}) \cup \{c \notin s : s \in S\} \vdash \neg\chi(m_1, c),$$

which we readily rewrite to

$$\mathrm{Diag}_{\mathrm{el}}(\mathcal{M}) \models \chi(m_1, c) \Rightarrow \bigvee_{s \in S} c \in s,$$

by choosing $m_2 = (s_1 \cup \ldots \cup s_n)^{\mathcal{M}} \in M$, we have proved the claim. Again since $\varphi(v)$ isolates $p_a(v)$, we also have:

(1) $T \vdash (\forall v)(\varphi(v) \Rightarrow v \in a)$.

(2) $T \cup \{\varphi(v)\}$ is satisfiable.

By (2), $\chi(m_v, c)$ is true in an elementary extension of $\mathcal{M}$ satisfying T (for a well-chosen element m_v).
For any $m \in M$, we have by (1) and $c \notin m \in T$, that the sentence

$$(\exists y \notin m)(\exists v \in a)\chi(v, y)$$

is true in this elementary extension of $\mathcal{M}$, and by consequence also in $\mathcal{M}$. Thus,

$$\mathcal{M} \models (\forall x)(\exists y \notin x)(\exists v \in a)\chi(v, y).$$

This means that

$$\mathcal{M} \models \{y : (\exists v \in a)\chi(v, y)\} \text{ is a proper class.}$$

Since a proper class can not be written as a set-size union of sets, we find $m_1 \in M$ such that

$$\mathcal{M} \models (\forall x)(\exists y \notin x)\ \chi(m_1, y).$$

Our claim learns

$$\mathcal{M} \models (\exists x_2)(\forall y)(\chi(m_1, y) \Rightarrow y \in x_2.)$$

Both statements are clearly contradictory.
This contradiction proves that the types $p_a(v)$ with $a \in M$ are in fact non-isolated and hence concludes the proof. □

References and Remarks

The proofs of Theorems 6.1 and 6.4 given here are based on the corresponding proofs in respectively Marker (2002) and Chang and Keisler (1990). The proof of Theorem 6.3 is based on the corresponding proof in Marker (2002).

7

The MacDowell-Specker Theorem

7.1 The MacDowell-Specker Theorem for PA-models

The omitting-types-proof of Theorem 6.1 goes wrong for $\mathcal{M} \models \mathsf{PA}$ that are uncountable, because in this case the language $\mathcal{L}_{\mathcal{M}}$ becomes uncountable (just as the set of types to be omitted). Examples such as the following one show that there is no refuge in attempting to generalize the Omitting Types Theorem to uncountable languages $\mathcal{L}$.

Example 7.1
Let $\mathcal{L}$ be the (uncountable) language consisting of a set of constant symbols $C \sqcup D$, where $\aleph_0 = |C| < |D|$.
Consider the $\mathcal{L}$-theory

$$T = \{d_1 \neq d_2 : d_1, d_2 \in D\}$$

and the 1-type

$$p(v) = \{c \neq v : c \in C\}.$$

Then, we argue, $p(v)$ is non-isolated.
Suppose $\varphi(v)$ is an $\mathcal{L}$-formula such that $\mathcal{N} \models \varphi(a)$, for a T-model $\mathcal{N}$ and $a \in N$. Since $\varphi(v)$ contains only finitely many constants, we can choose $c \in C$ not occurring in $\varphi(v)$. By putting $c^{\mathcal{N}} = a$ and leaving the rest of $\mathcal{N}$ unchanged, we have found a model of $T \cup \{\neg(\forall v \varphi(v) \Rightarrow c \neq v)\}$, proving that $p(v)$ is non-isolated.
The conclusion of the Omitting Types Theorem is however false in this case, since any model $\mathcal{N}$ of T will be uncountable and therefore

$$\{d^{\mathcal{N}} : d \in D\} \setminus \{c^{\mathcal{N}} : c \in C\} \neq \emptyset,$$

but any element of $\{d^{\mathcal{N}} : d \in D\} \setminus \{c^{\mathcal{N}} : c \in C\}$ will realise the type $p(v)$.

It is therefore quite surprising that the following theorem that was already announced in the previous chapter holds.

Theorem 7.1 (MacDowell-Specker (1961))
Every $\mathcal{M} \models \mathsf{PA}$ has an elementary end extension.

The original proof of the MacDowell-Specker Theorem made use of an ultraproduct construction. However, it has become more fashionable[1] to prove Theorem 7.1 as a corollary to a slightly stronger theorem due to Gaifman (Gaifman (1976)) and Phillips (Phillips (1974)).

Theorem 7.2 (Gaifman (1976), Phillips (1974))
Every $\mathcal{M} \models \mathsf{PA}$ has a conservative elementary extension.

We will follow fashion and prove Theorem 7.2.
First, some necessary definitions.

Definition 7.1
Let $\mathcal{M}$ be an $\mathcal{L}$-structure. A set $S \subseteq M^n$ is *definable* in $\mathcal{M}$, if there is an $\mathcal{L}_{\mathcal{M}}$-formula $\varphi(\overline{v})$ such that

$$S = \{\overline{x} \in M^n : \mathcal{M} \models \varphi(\overline{x})\}.$$

If $\mathcal{N}$ is an extension of the $\mathcal{L}$-structure $\mathcal{M}$, then $\mathcal{N}$ is called *conservative* if for any set $S \subseteq N$ that is definable in $\mathcal{N}$, the set $S \cap M$ is definable in $\mathcal{M}$.

Let us now understand why Gaifman's result implies Theorem 7.1.
This is because, when it comes to models of PA, conservative extensions are also end extensions[2].

Lemma 7.1
Let $\mathcal{M}, \mathcal{N}$ be PA-models, and suppose that $\mathcal{N}$ is a conservative extension of $\mathcal{M}$, then $\mathcal{N}$ is also an end extension of $\mathcal{M}$.

Proof
Let $n \in N \setminus M$. We need to show that for any $m \in M$, $\mathcal{N} \models n > m$.
The set $D_n = \{m \in \mathcal{N} : \mathcal{N} \models m < n\}$ is definable in $\mathcal{N}$ and therefore

$$D_n \cap M = \{m \in M : \mathcal{N} \models m < n\}$$

is definable in $\mathcal{M}$. Then $D_n \cap M$ is susceptible to the induction principle in $\mathcal{M}$ and since $0 \in D_n \cap M$ and

$$(\forall x \in D_n \cap M)(\mathcal{M} \models x + 1 \in D_n \cap M),$$

we have $D_n \cap M = M$. □

Conservative extensions can be constructed starting from a special species of types.

Definition 7.2
Let $\mathcal{M}$ be an $\mathcal{L}$-structure.
A complete $\mathcal{M}$-type $p(\overline{v}) \in S_n(\mathcal{M})$ is *definable* if for every $\mathcal{L}$-formula $\varphi(\overline{v}, \overline{w})$, the set $\{\overline{m} \in M^k : \varphi(\overline{v}, \overline{m}) \in p(\overline{v})\} \subseteq M^k$ is definable over $\mathcal{M}$.

[1] At least, this is the approach followed by the two standard works Kaye (1991) and Kossak and Schmerl (2006).
[2] As we will see in a minute, this is *not* true for ZFC-models.

Definition 7.3
An $\mathcal{L}$-theory T has *definable Skolem functions* if for any $\mathcal{L}$-formula $\varphi(\overline{w}, v)$, there is a formula $\psi(\overline{w}, v)$ such that in any T-model $\mathcal{M}$,

- the set $\{(\overline{x}, y) \in M^{n+1} : \mathcal{M} \models \psi(\overline{x}, y)\}$ is a total function $M^n \to M$,
- $\mathcal{M} \models (\forall \overline{w})(\exists v \varphi(\overline{w}, v) \Rightarrow \exists u(\psi(\overline{w}, u) \wedge \varphi(\overline{w}, u)))$.

We say that an $\mathcal{L}$-structure $\mathcal{M}$ has definable Skolem functions when the $\mathcal{L}_{\mathcal{M}}$-theory $\mathrm{Diag}_{\mathrm{el}}$ has definable Skolem functions.

Definition 7.4
Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$. Then $m \in M$ is *definable* in $\mathcal{M}$ over A if there is an $\mathcal{L}$-sentence $\varphi(v, \overline{w})$ and $\overline{a} \in A^n$ such that $\mathcal{M} \models \varphi(m, \overline{a}) \wedge (\exists! v)\varphi(v, \overline{a})$. We define the definable closure of A in $\mathcal{M}$

$$\mathrm{Cl}(A, \mathcal{M}) = \{m \in M : m \text{ is definable in } \mathcal{M} \text{ over } A\}.$$

It is clear that $\mathrm{Cl}(A, \mathcal{M})$ will always determine a sub-structure of $\mathcal{M}$, for $\mathrm{Cl}(A, \mathcal{M})$ is closed under the interpretation $f^{\mathcal{M}}$ of function symbols and contains all $c^{\mathcal{M}}$, where c is a constant symbol of $\mathcal{L}$. In case $\mathcal{M}$ has definable Skolem functions, we have even more.

Lemma 7.2
If the $\mathcal{L}$-structure $\mathcal{M}$ has definable Skolem functions, then for any $A \subseteq M$, the definable closure of A in M is an elementary sub-structure of $\mathcal{M}$:

$$\mathrm{Cl}(A, \mathcal{M}) \prec \mathcal{M}.$$

Proof
We invoke the Tarski-Vaught-criterion. Because $\mathrm{Cl}(A, \mathcal{M})$ is a sub-structure of $\mathcal{M}$, it suffices to prove for any $\mathcal{L}$-formula $\varphi(\overline{w}, v)$, for any $\overline{d} \in \mathrm{Cl}(A, M)$, that if $\mathcal{M} \models (\exists v)\varphi(\overline{d}, v)$, then there is $e \in \mathrm{Cl}(A, \mathcal{M})$ such that $\mathcal{M} \models \varphi(\overline{d}, e)$.
Let $\overline{d} = (d_1, \ldots, d_l)$ and let $\chi_i(v, \overline{a}_i)$ define d_i in $\mathcal{M}$. We consider the formula

$$\varphi'(\overline{a}_1, \ldots, \overline{a}_l, v) \equiv (\exists v_1 \ldots, v_l)(\varphi(v_1, \ldots, v_l, v) \wedge \bigwedge_{1 \le i \le l} \chi_i(v, \overline{a}_i).$$

Because $\mathcal{M}$ has definable Skolem functions, we can choose $\psi(\overline{w}, v)$ as in Definition 7.3, defining a Skolem function for φ'. This implies that there is $e \in M$ such that $\mathcal{M} \models \psi(\overline{a}_1, \ldots, \overline{a}_l, e)$ and moreover, e is defined in $\mathcal{M}$ by $\psi(\overline{a}_1, \ldots, \overline{a}_l, v)$, hence $e \in \mathrm{Cl}(A, \mathcal{M})$. Because ψ defines a Skolem function for φ', we also find $\mathcal{M} \models \varphi'(\overline{a}_1, \ldots, \overline{a}_l, e)$, from which it follows that also $\mathcal{M} \models \varphi(\overline{d}, e)$, as required. □

It is now clear that in fact, $\mathrm{Cl}(A, \mathcal{M})$ is the smallest elementary sub-structure of $\mathcal{M}$ that contains A.

Theorem 7.3
Let T be a complete $\mathcal{L}$-theory with definable Skolem functions. Let $\mathcal{M}$, $\mathcal{N}$ be two T-models. Then there is a unique elementary embedding $\mathrm{Cl}(\emptyset, \mathcal{M}) \hookrightarrow \mathcal{N}$ and the image of this embedding is $\mathrm{Cl}(\emptyset, \mathcal{N})$.

Proof
By Lemma 7.2, $\mathrm{Cl}(\emptyset, \mathcal{M}) \prec \mathcal{M} \models T$, so $\mathrm{Cl}(\emptyset, \mathcal{M}) \models T$.
Then for any $\mathcal{L}$-formula $\chi(x)$,

$$\begin{aligned}\mathcal{N} \models \exists! x\chi(x) &\iff T \vdash \exists! x\chi(x)\\ &\iff \mathrm{Cl}(\emptyset, \mathcal{M}) \models \exists! x\chi(x).\end{aligned}$$

Choose for each $a \in \mathrm{Cl}(\emptyset, \mathcal{M})$ a formula $\chi_a(x)$ defining a in $\mathcal{M}$.
Define $\iota : \mathrm{Cl}(\emptyset, \mathcal{M}) \hookrightarrow \mathcal{N}$. For each $x \in \mathrm{Cl}(\emptyset, \mathcal{M})$ let $\iota(x)$ be the unique element in N such that

$$\mathcal{N} \models \chi_x(\iota(x)).$$

It is immediate that any elementary embedding $\mathrm{Cl}(\emptyset, \mathcal{M}) \hookrightarrow \mathcal{N}$ should be of this form.
This is an elementary embedding: if $\mathrm{Cl}(\emptyset, \mathcal{M}) \models \varphi(x_1, \ldots, x_m)$, then

$$\begin{aligned}\mathrm{Cl}(\emptyset, \mathcal{M}) &\models \exists\overline{v}\varphi(v_1, \ldots, v_m) \wedge \chi_{x_1}(v_1) \wedge \ldots \wedge \chi_{x_m}(v_m)\\ &\Rightarrow T \models \exists\overline{v}\varphi(v_1, \ldots, v_m) \wedge \chi_{x_1}(v_1) \wedge \ldots \wedge \chi_{x_m}(v_m)\\ &\Rightarrow \mathcal{N} \models \exists\overline{v}\varphi(v_1, \ldots, v_m) \wedge \chi_{x_1}(v_1) \wedge \ldots \wedge \chi_{x_m}(v_m)\\ &\Rightarrow \mathcal{N} \models \varphi(\iota(x_1), \ldots, \iota(x_m)).\end{aligned}$$

Since $\iota(x)$ is definable in $\mathcal{N}$ by $\chi_x(v)$, $\iota[\mathrm{Cl}(\emptyset, \mathcal{M})] \subseteq \mathrm{Cl}(\emptyset, \mathcal{N})$.
Conversely, if y is definable in $\mathcal{N}$ by the formula $\varphi(v)$, then also

$$\mathrm{Cl}(\emptyset, \mathcal{M}) \models (\exists! v)\varphi(v).$$

Suppose that for $x \in \mathrm{Cl}(\emptyset, \mathcal{M})$, $\mathrm{Cl}(\emptyset, \mathcal{M}) \models \varphi(x)$.
Then $\mathrm{Cl}(\emptyset, \mathcal{M}) \models (\exists v)\varphi(v) \wedge \chi_x(v)$, but then also $\mathcal{N} \models (\exists v)\varphi(v) \wedge \chi_x(v)$, and we get $\mathcal{N} \models \chi_x(y)$, so $y = \iota(x)$. □

Definition 7.5
Let $\mathcal{M}$ be an $\mathcal{L}$-structure with definable Skolem functions.
If $p(v) \in S_1(\mathcal{M})$ is a complete $\mathcal{M}$-type, then $\mathcal{M}(p)$ is the $\mathcal{L}$-structure obtained by choosing an elementary extension $\mathcal{N}$ of $\mathcal{M}$ satisfying the complete $\mathcal{L}_{\mathcal{M}} \cup \{c\}$-theory $\{\varphi(c) : \varphi(v) \in p(v)\} \cup \mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$, where c is an extra constant and then setting

$$\mathcal{M}(p) := \mathrm{Cl}(\mathcal{M} \cup c^{\mathcal{N}}, \mathcal{N})$$

(where we induce a reduction, to obtain again an $\mathcal{L}$-structure).

By Theorem 7.3, $\mathcal{M}(p)$ is well-defined (up to isomorphism) and does not depend on the chosen elementary extension $\mathcal{N}$.

We can now explain how definable types give rise to conservative extensions.

Lemma 7.3
Let $\mathcal{M}$ be an $\mathcal{L}$-structure with definable Skolem functions.
For every definable type $p(v) \in S_1(\mathcal{M})$, $\mathcal{M}(p)$ is a conservative extension of $\mathcal{M}$.

Proof
Let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$, where $c \in N$ realises the type $p(v)$.
We have to show that $\mathrm{Cl}(\mathcal{M} \cup \{c\}, \mathcal{N})$ is a conservative extension of $\mathcal{M}$.
Because $\mathrm{Cl}(\mathcal{M} \cup \{c\}, \mathcal{N}) \prec \mathcal{N}$, c realises $p(v)$ in $\mathrm{Cl}(M \cup \{c\}, \mathcal{N})$.

Let $S \subseteq \mathrm{Cl}(M \cup \{c\}, \mathcal{N})$ be a definable subset of $\mathrm{Cl}(M \cup \{c\}, \mathcal{N})$, i.e. we can write

$$S = \{x \in \mathrm{Cl}(M \cup \{c\}, \mathcal{N}) : \mathrm{Cl}(M \cup \{c\}, \mathcal{N}) \models \varphi(x, \overline{a})\},$$

with $\overline{a} \in \mathrm{Cl}(M \cup \{c\})$. Since all elements in $\mathrm{Cl}(M \cup \{c\})$ are actually definable over $M \cup \{c\}$, we do not loose generality by writing $\varphi(x, \overline{a}) = \psi(x, c, \overline{b})$, with $\overline{b} \in M$. Then

$$\begin{aligned} S \cap M &= \{x \in M : \mathrm{Cl}(M \cup \{c\}, \mathcal{N}) \models \psi(x, c, \overline{b})\} \\ &= \{x \in M : \psi(x, v, \overline{b}) \in p(v)\}. \end{aligned}$$

The last set is definable in $\mathcal{M}$ because the type $p(v)$ is definable. □

Our last ingredient for the proof of the Gaifman-Philips-Theorem is the Coherence Principle COH.

Definition 7.6
We will say that the Coherence Principle COH holds in an $\mathcal{L}_A$-structure $\mathcal{M}$ if for any unbounded definable subset D of M and any $\mathcal{L}_{\mathcal{M}}$-formula $\varphi(v_1, v_2)$, there is an unbounded definable subset $E \subseteq D$ such that

$$\begin{aligned} \mathcal{M} \models \forall v_1 (&\exists w \forall v_2 > w (v_2 \in E \Rightarrow \varphi(v_1, v_2))) \vee \\ &(\exists w \forall v_2 > w (v_2 \in E \Rightarrow \neg\varphi(v_1, v_2))). \end{aligned}$$

Lemma 7.4
COH holds for any model $\mathcal{M}$ of PA.

Proof
We reason inside the model $\mathcal{M}$. Fix a coding that will allow us to speak about finite sets of M. Let D and $\varphi(v_1, v_2)$ be as in Definition 7.6.
We will choose $E = \{x_i : i \in M\}$, where $(x_i)_i$ is constructed by recursion.
Along, we construct finite sets $(I_i)_i$, $(J_i)_i \subseteq \{0, \ldots, i\}$.
It is convenient to write, for $i \in M$,

$$X_i = \{x \in M : \varphi(i, x)\}$$

and

$$Y_i = X_i^c = \{x \in M : \neg\varphi(i, x)\}.$$

Let $I_0 = \{0\}$, $J_0 = \emptyset$, $x_0 = \min X_0 \cap D$ if $X_0 \cap D$ is unbounded,
and $I_0 = \emptyset$, $J_0 = \{0\}$, $x_0 = \min Y_0 \cap D$ otherwise.
We define x_{i+1} as follows.
By induction we can assume that $S_i = \bigcap_{j \in I_i} X_j \cap \bigcap_{j \in J_i} Y_j \cap D$ is unbounded.
Choose $I_{i+1} = I_i \cup \{i\}$, $J_{i+1} = J_i$, $x_{i+1} = \min S_i \cap X_i \setminus \{x_0, \ldots, x_i\}$ if $S_i \cap X_i$ is unbounded,
and $I_{i+1} = I_i$, $J_{i+1} = J_i \cup \{i\}$, $x_{i+1} = \min S_i \cap Y_i \setminus \{x_0, \ldots, x_i\}$ otherwise.
It follows from this construction that

$$\mathcal{M} = \bigcup_{i \in M} I_i \sqcup \bigcup_{i \in M} J_i$$

and

$$\bigcup_{i\in M} I_i \subseteq \{i \in M : (\forall j > i)(x_j \in X_i)\},$$

$$\bigcup_{i\in M} J_i \subseteq \{i \in M : (\forall j > i)(x_j \in Y_i)\},$$

hence $E = \{x_i : i \in M\}$ has the required properties. □

Definition 7.7
We will call a 1-type $p(v)$ of a PA-model $\mathcal{M}$ *unbounded* if for every $a \in M$, the formula $a < v$ is contained in $p(v)$.

This is all we need to write down a proof for Theorem 7.2.

Lemma 7.5
Every PA-model $\mathcal{M}$ has an unbounded definable type $p(v)$.

Proof
Let $(\varphi_i(v_1, v_2))_{i<\omega}$ be an enumeration of all $\mathcal{L}_A$-formulas with two free variables v_1 and v_2.
By a meta-recursion, we construct a decreasing sequence $(S_i)_{i<\omega}$ of unbounded subsets of M that are definable in $\mathcal{M}$ and such that for every $i < \omega$,

$$\begin{aligned}\mathcal{M} \models \forall v_1((\exists w \forall v_2 > w(v_2 \in S_{i+1} \Rightarrow \varphi_i(v_1, v_2)))\vee \\ (\exists w \forall v_2 > w(v_2 \in S_{i+1} \Rightarrow \neg\varphi_i(v_1, v_2)))).\end{aligned}$$

This construction is possible because the coherence principle COH holds in $\mathcal{M}$.
Choose formulas $(\sigma_i)_{i<\omega}$ in $\mathcal{L}_{\mathcal{M}}$ such that $S_i = \{m \in M : \mathcal{M} \models \sigma_i(m)\}$.
Let $t_0(v)$ be the type

$$\{v > a : a \in M\} \cup \mathrm{Diag}_{\mathrm{el}}(\mathcal{M}) \cup \{\sigma_i(v) : i < \omega\}$$

and let $t(v)$ be the 1-type

$$\{\mathcal{L}_{\mathcal{M}}\text{-formulas } \varphi \text{ with free variables contained in } \{v\} \text{ such that } t_0(v) \vdash \varphi\}.$$

By compactness, $t(v)$ is consistent.
We check that $t(v)$ is complete and conservative. Let $\varphi(v)$ be an arbitrary $\mathcal{L}_{\mathcal{M}}$-formula. Then, without loss of generality, $\varphi(v) \equiv \varphi_i(a, v)$ for certain $i < \omega$ and $a \in M$. It follows from the construction of $(S_i)_{i<\omega}$ that either of the following holds:

$$\mathcal{M} \models (\exists w \forall v > w)\sigma_{i+1}(v) \Rightarrow \varphi(v)$$

or

$$\mathcal{M} \models (\exists w \forall v > w)\sigma_{i+1}(v) \Rightarrow \neg\varphi(v).$$

Since $t(v) \vdash v > m_w \wedge \sigma_{i+1}(v)$, for any $m_w \in M$, we get in the first case

$$\varphi(v) \in t(v)$$

and in the second case

$$\neg\varphi(v) \in t(v).$$

Hence, $t(v)$ is complete. Since $t(v)$ is in addition consistent, we find that for any $\mathcal{L}$-formula $\varphi_i(v_1, v_2)$, the set

$$\{m \in M : \varphi_i(m, v) \in p(v)\} = \{m \in M : \mathcal{M} \models (\exists w \forall v > w)\sigma_{i+1}(v) \Rightarrow \varphi_i(m, v)\}$$

is definable. It follows that $t(v)$ is conservative. □

Proof of Theorem 7.2
This follows from combining Lemma 7.3 and Lemma 7.5. □

7.2 A MacDowell-Specker Theorem for models of set theory

Surprisingly, both the MacDowell-Specker Theorem and the Gaifman-Philips Theorem can fail for models of ZFC. To illustrate this, we list without proof the following classic results.

Theorem 7.4 (Keisler and Silver (1970))
Let κ be an inaccessible cardinal. If the cardinality of the set of inaccessibles below κ is strictly smaller than κ, then the ZFC-model $\mathcal{M} = (V_\kappa, \in^{\mathcal{M}} = \in\restriction V_\kappa)$ has no elementary end extensions.

Moreover, we can not regain the MacDowell-Specker Theorem by strengthening the theory ZF. This follows by the following surprisingly general theorem due to Matt Kaufmann.

Theorem 7.5 (Kaufmann (1983))
Any consistent extension T of ZFC has a model $\mathcal{M}$ of cardinality $\aleph_1$ which has no elementary end extensions.

For conservative elementary end extensions, the situation is even worse:

Theorem 7.6 (Enayat (1999))
Let $\mathcal{M} \models$ ZFC, then $\mathcal{M}$ has no conservative elementary end extensions.

It is therefore quite surprising that the argument with definable-types and Skolem-closures that was tailored for models of arithmetic, is again of value for providing conservative rank extensions of *countable* models of the stronger class theory $\mathsf{GBC} + \mathrm{On} \rightarrow (\mathrm{On})^3_2$. We will now introduce this theory.
Unlike ZFC, the theory GB is formulated in a two-sorted language.[3] GB distinguishes two sorts of objects: sets (typically denoted by lower case letters) and classes (typically denoted by capital letters). The following two special rules govern these sorts:

- Every set is a class.

[3] GB is therefore not a set theory in the strict sense we adopted at the beginning of Chapter 1, this is why we opt for the term *class theory* instead.

- Every element of a class is a set.

The set axioms of pairing, infinity, union, powerset and regularity are formulated in the usual way. The axiom of extensionality now talks about all classes:

Axiom of Extensionality for Classes

$$\forall X, Y \quad \forall u(u \in X \Leftrightarrow u \in Y) \Rightarrow (X = Y).$$

Lastly, the separation axiom and the replacement axiom are strengthened to allow for classes featuring in the formula φ.
The theory GBC includes the following axiom of global choice.

Axiom of Global Choice

There is a class function $\text{Choice} : V \to V$ (the global choice function) such that
$$(\forall x \neq \emptyset)(\text{Choice}(x) \in x).$$

The theory GBC will now play the role of analogue for the theory PA. However, PA proves strong Ramsey-like properties (such as the principle COH), that did come up in the proof of Lemma 7.5, but our not matched by GBC. This is why we need one more axiom.

$\mathbf{On} \to (\mathbf{On})^3_2$

This axiom asserts that for any class function $F : [\text{On}]^3 \to 2$, there is a proper class $Y \subseteq \text{On}$ that is homogeneous for F.

Theorem 7.7 (McAloon and Ressayre (1981))
Let $\mathcal{M}^* = (M, C, \in)$ be a countable model of the theory GBC+$\text{On} \to (\text{On})^3_2$. Let $\mathcal{M} = (M, \in, (\underline{X}^{\mathcal{M}})_{X \in C})$ be the corresponding $\mathcal{L}_\in(C)$ structure, where $\mathcal{L}_\in(C)$ is the language obtained by adding to $\mathcal{L}_\in$ a 1-ary predicate symbol $\underline{X}$ for any $\mathcal{M}^*$-class X (with obvious interpretations $\underline{X}^{\mathcal{M}} = X$). There exists an $\mathcal{L}_\in(C)$-structure $\mathcal{N} = (N, \in, (\underline{X}^{\mathcal{N}})_{X \in C})$ such that $\mathcal{N}$ is an elementary and conservative rank extension of $\mathcal{M}$.

Proof
Let $(F_n)_{n<\omega}$ be an enumeration of all class functions $F : [M]^3 \to 2$ in $\mathcal{M}^*$.
Let's call a class function $G : X \to V$ in $\mathcal{M}^*$ bounded if

$$\mathcal{M}^* \models G[X] \text{ is a set.}$$

Let $(G_n)_{n<\omega}$ be an enumeration of all bounded class functions $G : \text{On} \to V$ in $\mathcal{M}^*$. Use $\mathcal{M}^* \models \text{On} \to (\text{On})^3_2$ and $\mathcal{M}^* \models \text{On} \to (\text{On})^1_\kappa$ recursively to find a sequence $(X_n)_{n<\omega}$ of proper classes in $\mathcal{M}^*$ with the properties

- $\mathcal{M}^* \models X_n \subseteq \text{On}$, for all $n < \omega$,
- $\mathcal{M}^* \models X_{n+1} \subseteq X_n$,

- $\mathcal{M}^* \models X_n$ is F_n-homogeneous.
- $\mathcal{M}^* \models G_n$ is constant on X_n.

Let $t_0(v)$ be the type

$$\{v \in \mathrm{On}\} \cup \{v > \alpha : \alpha \in \mathrm{On}^{\mathcal{M}}\} \cup \mathrm{Diag}_{\mathrm{el}}(\mathcal{M}) \cup \{\underline{X}_n(v) : n < \omega\}$$

(with $\mathrm{Diag}_{\mathrm{el}}(\mathcal{M})$ with respect to the language $\mathcal{L}_{\in}(C)$) , and let $t(v)$ be the type

$$\{\mathcal{L}_{\in}(C)\text{-formulas } \varphi \text{ with free variables contained in } \{v\} \text{ such that } t_0(v) \vdash \varphi\}.$$

Both are satisfiable by compactness. We prove that $t(v)$ is a complete conservative type over $\mathcal{M}$.
Let $(\varphi_n(v_1, v_2))_{n<\omega}$ be an enumeration of all $\mathcal{L}_{\in}(C)$-formulas in the two variables v_1, v_2. We recursively construct a subsequence $(Y_n)_{n<\omega}$ of $(X_n)_{n<\omega}$ with the property that for any $n < \omega$ and $\alpha \in \mathrm{On}^{\mathcal{M}}$ either

$$\mathcal{M} \models (\exists\beta)(\forall v > \beta)(\underline{X}_{n+1}(v) \Rightarrow \varphi_n(\alpha, v))$$

or

$$\mathcal{M} \models (\exists\beta)(\forall v > \beta)(\underline{X}_{n+1}(v) \Rightarrow \neg\varphi_n(\alpha, v)). \tag{7.1}$$

Construction:
$Y_0 := X_0$. Let Y_n be defined.
Consider the following class function $F : [X_n]^3 \to 2$, defined by

$$\forall\{\alpha, \beta, \gamma\} \subseteq X_n \text{ with } \alpha < \beta < \gamma,$$
$$F(\alpha, \beta, \gamma) = 0 \iff (\forall u < \alpha)(\varphi_n(u, \beta) \iff \varphi_n(u, \gamma)).$$

There is $m < \omega$ such that F is the restriction of $F_m : [M]^3 \to 2$ to $[X_n]^3$. It follows that there is $k \geq n$ such that X_k is F-homogeneous. Let $Y_{n+1} = X_k$.
Claim: $(\forall\alpha \in Y_{n+1})(\exists\beta, \gamma \in Y_{n+1})(F(\alpha, \beta, \gamma) = 0)$.
To prove this claim, consider $G : Y_{n+1} \backslash (\alpha+1) \to P(\alpha) : \beta \mapsto \{u < \alpha : \varphi_n(u, \beta)\}$.
Since Y_{n+1} is a proper class, G can not be injective and the claim follows.
Since Y_{n+1} is F-homogeneous, it follows from this claim that F is constantly 0 on $[Y_{n+1}]^3$. So Y_{n+1} has the following feature:

$$(\forall\alpha, \beta, \gamma \in Y_{n+1} \text{ with } \alpha < \beta < \gamma)(\forall u < \alpha)(\varphi_n(u, \beta) \iff \varphi_n(u, \gamma)). \tag{7.2}$$

Checking equation (7.1):
Choose $\alpha \in \mathrm{On}^{\mathcal{M}}$ arbitrarily. There exists $\alpha' < \beta$ both in Y_{n+1}, with $\alpha < \alpha'$. It follows from equation (7.2) that $\forall\gamma \in Y_{n+1}$, with $\gamma > \beta$, $\varphi_n(\alpha, \beta) \iff \varphi_n(\alpha, \gamma)$.
Hence either $\mathcal{M} \models (\forall v > \beta)(\underline{Y}_{n+1}(v) \Rightarrow \varphi_n(\alpha, v))$ (in case $\varphi_n(\alpha, \beta)$) or
$\mathcal{M} \models (\forall v > \beta)(\underline{Y}_{n+1}(v) \Rightarrow \neg\varphi_n(\alpha, v))$ (in case $\neg\varphi_n(\alpha, \beta)$).
It follows that for each $n < \omega$ and each $\alpha \in \mathrm{On}^{\mathcal{M}}$ the following are equivalent:

$$\begin{aligned}
&\mathcal{M} \models \exists\beta(\forall v > \beta)(\underline{Y}_{n+1}(v) \Rightarrow \varphi_n(\alpha, v))\\
&\quad\iff \varphi_n(\alpha, v) \in t(v)\\
&\quad\iff \mathcal{M} \not\models \exists\beta(\forall v > \beta)(\underline{Y}_{n+1}(v) \Rightarrow \neg\varphi_n(\alpha, v))\\
&\quad\iff \neg\varphi_n(\alpha, v) \notin t(v).
\end{aligned} \tag{7.3}$$

To check that $t(v)$ is complete, let $\varphi(m,v)$ be an arbitrary $\mathcal{L}_\in(C)$-formula with $m \in M$ (it suffices again to consider only formulas with one parameter). Since

$$\mathcal{M}^* \models \textsf{Global Choice},$$

we find class functions $B : V \to \mathrm{On}$ and $B^{-1} : \mathrm{On} \to V$ that are inverses.
Let $\varphi'(w,v) = \varphi(B^{-1}(w), v)$ and let $\alpha = B(m)$. Then

$$\begin{aligned} \varphi(m,v) \in t(v) &\iff \varphi'(\alpha,v) \in t(v) \\ &\overset{(7.3)}{\iff} \neg\varphi'(\alpha,v) \notin t(v) \\ &\iff \neg\varphi(m,v) \notin t(v). \end{aligned}$$

Moreover,

$$\begin{aligned} &\{m \in M : \varphi(m,v) \in t(v)\} \\ &\qquad = \{m \in M : \mathcal{M} \models \exists(\alpha,\beta)(\forall v > \beta)(\underline{Y}_{n+1}(v) \Rightarrow \varphi_n(\alpha,v)) \wedge B(m) = \alpha\} \end{aligned}$$

(for certain $n < \omega$) and hence $t(v) \in S_1(M)$ is a definable type.
Choose an elementary extension $\mathcal{A} = (A, \in, (\underline{X}^{\mathcal{A}})_{X \in C})$ of $\mathcal{M}$, with $c \in A$ realising the type $t(v)$. Since the global choice function of $\mathcal{M}$ gets again interpreted as a global choice function in $\mathcal{A}$, $\mathcal{A}$ has definable Skolem functions. It follows that $\mathcal{N} = \mathrm{Cl}(M \cup \{c\}, \mathcal{A})$, the definable closure of $M \cup \{c\}$ in $\mathcal{A}$ is an elementary substructure of $\mathcal{A}$, but then also an elementary extension of $\mathcal{M}$. It is also clear that c realises the type $t(v)$ in $\mathcal{N}$. Since $t(v)$ was definable, $\mathcal{N}$ is a conservative extension of $\mathcal{M}$.
We still need to check that $\mathcal{N}$ is a rank extension.
Let $x \in M$ and suppose $a \in A$ is definable over $M \cup \{c\}$. We have to show that if $\mathcal{A} \models a \in x$, then $a \in M$. Let $\varphi(v, \overline{m}, c)$ be a definition of a over $M \cup \{c\}$ in $\mathcal{A}$, i.e.:

$$\mathcal{A} \models (\exists! x)(\varphi(x, \overline{m}, c)) \quad \text{and} \quad \mathcal{A} \models \varphi(a, \overline{m}, c).$$

There is $n < \omega$ such that

$$\mathcal{M} \models (\forall y)(G_n(y) = \mathrm{Choice}(\{x : \varphi(x, \overline{m}, b)\})).$$

By construction,

$$\mathcal{M} \models G_n \text{ is constant on } X_n.$$

So for certain $d \in M$,

$$\mathcal{M} \models (\forall x)(X_n(x) \Rightarrow G_n(x) = d).$$

By $\mathcal{A} \models \underline{X_n}(c)$ and $\mathcal{M} \prec \mathcal{A}$, we have $\mathcal{A} \models G_n(c) = d$.
Then $\mathcal{A} \models d \in \{x : \varphi(x, \overline{m}, c)\}$, but since the formula $\varphi(v, \overline{m}, c)$ defines a over $\mathcal{A}$, we infer

$$a = d \in M.$$

□

References and Remarks

Example 7.1 can be found in Chang and Keisler (1990).
The proof of the Gaifman-Phillips Theorem 7.2 as presented here is based on the one presented in Wong (2014). We thank Ali Enayat for supplying a vital step in the proof of Theorem 7.7, which seems to have been overlooked by the authors of McAloon and Ressayre (1981).
We wish to make a further remark on the proof of Theorem 7.7. Upon closer inspection of the given proof, one finds that $\mathrm{On} \to (\mathrm{On})^3_2$ is actually used to prove the following set theoretical version of the coherence principle.
Coherence principle COH in set theory:
For any class function $R : \mathrm{On}^2 \to 2$, there exists a proper class $Y \subseteq \mathrm{On}$ such that for any $\alpha \in \mathrm{On}$:
either

$$(\exists \beta \in \mathrm{On})(Y \setminus \beta \subseteq \{\gamma : R(\alpha, \gamma) = 0\})$$

or

$$(\exists \beta \in \mathrm{On})(Y \setminus \beta \subseteq \{\gamma : R(\alpha, \gamma) = 1\}).$$

The following observation is part of the proof:

Lemma 7.6
COH is provable from $\mathrm{On} \to (\mathrm{On})^3_2$ over GBC.

Proof
Define $F : [\mathrm{On}]^3 \to 2$ by $(\forall \{\alpha, \beta, \gamma\} \subseteq [\mathrm{On}]^3)$ with $\alpha < \beta < \gamma$,

$$F(\alpha, \beta, \gamma) = 0 \iff (\forall u < \alpha)(R(u, \beta) = R(u, \gamma)).$$

Choose $Y \subseteq \mathrm{On}$ a proper class that is F-homogeneous.
Claim: $(\forall \alpha \in Y)(\exists \beta, \gamma \in Y)(F(\alpha, \beta, \gamma) = 0)$.
To prove this claim, consider

$$G : Y \to P(\alpha) : \beta \mapsto \{u < \alpha : R(u, \beta) = 0\}.$$

Since Y is a proper class, G can not be injective and the claim follows.
Since Y is F-homogeneous, it follows from this claim that F is constantly 0 on $[Y]^3$. So Y has the following feature:

$$(\forall \alpha, \beta, \gamma \in Y \text{ with } \alpha < \beta < \gamma)(\forall u < \alpha)(R(u, \beta) = R(u, \gamma)).$$

It follows that Y is an R-cohesive set:
Choose $\alpha \in \mathrm{On}$ arbitrarily. There exists $\alpha' < \beta \in Y_{n+1}$ such that $\alpha < \alpha' < \beta$.
It follows that $\forall \gamma \in Y$ with $\gamma > \beta$, $R(\alpha, \beta) = R(\alpha, \gamma)$.
Hence, either
$(\forall v > \beta)(v \in Y \Rightarrow R(\alpha, v) = 0)$ (in case $R(\alpha, \beta) = 0$)
or
$(\forall v > \beta)(v \in Y \Rightarrow R(\alpha, v) = 1)$ (in case $R(\alpha, \beta) = 1$). □

Once this has been accomplished, the given proof of Theorem 7.7 is a the set-theoretic analogue of the given proof of the Gaifman-Phillips Theorem 7.2.

This led us to ask the question whether the combinatorial principle $\mathrm{On} \to (\mathrm{On})^3_2$ is over GBC strictly stronger than COH. This question may be seen as natural in view of the currently emerging hierarchy of reverse mathematics above GBC, see for example Gitman et al. (2017). In personal communication with the author, Ali Enayat solved this problem by proving that both principles $\mathrm{On} \to (\mathrm{On})^3_2$ and COH turn out to be equivalent over GBC.

It is interesting that in contrast, this situation is not mirrored by the one in the reverse mathematics of second order arithmetic: $\mathsf{RCA}_0 + \mathsf{RT}^3_2$ is here equivalent to ACA_0 (see Simpson (2009)), while it is remarked in Wong (2014) that $\mathsf{RCA}_0 + \mathsf{COH}$ is strictly weaker.

8

The method of indicators in set theory

The goal of this last chapter is to explain how the results in the previous chapter can be applied to extend the Paris-Kirby method of indicators to set theory. This idea was worked out in the article McAloon and Ressayre (1981) that only appeared in French and seems today to some extent forgotten. We will give in this chapter a quick overview of some interesting theorems that appear in the article McAloon and Ressayre (1981), with the intention of bringing them back to the attention of current research.

8.1 n-dense sets

One of the many concepts that has its origin back in the heydays of combinatorial independence results for PA in the 1970's, is that of an *n-dense* set.

Definition 8.1
Let $\mathcal{M}$ be a PA-model, and S a non-empty definable subset of M.
We define recursively:
S is 0-*dense* if $|S| \geq \min S + 4$.
S is $n+1$-*dense* if for all partitions $f : [S]^3 \to \{0,1\}$, there exists $T \subseteq S$ that is n-dense and homogeneous for f.

A first surprise is that this notion of n-dense sets can be transferred almost ad verbatim to give an interesting notion of α-dense set in models of set theory.

Definition 8.2
Let $\mathcal{M}$ be a ZFC-model, and $\emptyset \neq x \in M$.
We define recursively:
x is 0-*dense* if $|x| \geq \beth_{\min x}$.
x is γ-*dense* if for all partitions $f : [x]^3 \to \{0,1\}$ and for all $\alpha < \gamma$, there exists $y \subseteq x$ that is α-dense and homogeneous for f.

Here, $\beth_\alpha$ stands for the α-th $\beth$-number.

Definition 8.3

$$\Phi(\gamma) \equiv (\forall\alpha)(\exists\beta)([\alpha, \beta] \text{ is } \gamma\text{-dense}).$$

It testifies of the naturality of this concept, that the statement $(\forall\gamma)\Phi(\gamma)$ turns out to possess large cardinal strength!
In McAloon and Ressayre (1981) it is proved that the statement $(\forall\gamma)\Phi(\gamma)$ is (over the base theory ZFC) equivalent to the statement

$$\forall\alpha\exists\kappa(\kappa \text{ is } \alpha\text{-Mahlo}).$$

We will consider the weaker statement $(\forall n < \omega)\Phi(n)$ and show that it also leads to an incompleteness result for ZFC. This result is due to the pioneering article McAloon and Ressayre (1981), as well. We show first that $(\forall\gamma)\Phi(\gamma)$ is not stronger than the existence of On-many weakly compacts[1].

Lemma 8.1
If κ is weakly compact and $x \subseteq \kappa$ cofinal in κ, then x is α-dense for any ordinal α.

Proof
Let κ be weakly compact. We prove by induction on α:
Every cofinal subset x of κ is α-dense.
For $\alpha = 0$, it suffices to point out that $|x| = \kappa = \beth_\kappa > \beth_{\min x}$ for all $x \subseteq \kappa$ cofinal. For the inductionstep $\alpha > 0$, take $\gamma < \alpha$ arbitrary, $x \subseteq \kappa$ cofinal and $f : [x]^3 \to 2$.
Because κ is weakly compact (and hence $\kappa \to (\kappa)^3_2$), there is a homogeneous $y \subseteq x$, that is still cofinal in κ. By the induction hypothesis, y is γ-dense. □

As a striking application of Theorem 7.7, we will show that over GBC, the principle $\text{On} \to (\text{On})^3_2$ implies the existence of an n-dense set above any ordinal, where n is any (meta-theoretic) natural number.

Definition 8.4
When R is a relation on A, then for any $x \in A$, we write

$$x{\downarrow} = \{y \in A : yRx\}.$$

When $\mathcal{M} = (M, \in^{\mathcal{M}})$ is a ZFC-model and $x \in M$, we also use the notation $x_\in$ instead of $x{\downarrow}$, i.e.

$$x_\in = \{y \in M : \mathcal{M} \models y \in x\}.$$

Definition 8.5
Let $(L, <)$ be a linear order, we define the *well-founded part* of $(L, <)$ as follows:

$$\text{wfp}(L, <) = \{x \in L : (x{\downarrow}, <) \text{ is well-founded}\}.$$

Equivalently, $\text{wfp}(L, <)$ is the maximal transitive well-founded subset S of L, i.e. S is $\subseteq$-maximal with respect to:

- $(S, <)$ is well-founded.

[1] For an overview of the terminology on these large cardinal concepts, see for example our bachelor's project De Bondt (2016)

- $(\forall x \in L)(\forall y \in S)(x < y \Rightarrow x \in S)$.

Lemma 8.2
Let $(L, <)$ be a linear order, then for any $S \subseteq L$,

$$\mathrm{wfp}(L, <) \cap S \subseteq \mathrm{wfp}(S, <).$$

Proof
If $x \in \mathrm{wfp}(L, <) \cap S$, then $(\{y \in S : y < x\}, <)$ is well-founded, because so is $(x\downarrow, <)$. Hence $x \in \mathrm{wfp}(S, <)$. □

Definition 8.6
For any ZFC-model $\mathcal{M} = (M, \in^{\mathcal{M}})$, we define its *ordinal standard part* $\mathrm{osp}(\mathcal{M})$:

$$\mathrm{osp}(\mathcal{M}) = \mathrm{wfp}(\mathrm{On}^{\mathcal{M}}, \in^{\mathcal{M}}).$$

Obviously, if $\mathcal{M} \models \mathsf{ZFC}$, then for any $\alpha \in \mathrm{On}^{\mathcal{M}}$,

$$\mathcal{M} \models \alpha \text{ is well-ordered},$$

but this does not imply that $(\alpha\downarrow, \in^{\mathcal{M}})$ is well-ordered. Indeed, it may very well be that $\alpha\downarrow = \alpha_{\in}$ contains a strictly decreasing sequence which is not recognised by (read: definable in) $\mathcal{M}$.

We will often identify $\mathrm{osp}(\mathcal{M})$ with its ordinal order type α and the elements of $\mathrm{osp}(\mathcal{M})$ with the ordinals $\gamma < \alpha$.

Theorem 8.1
Let $\mathcal{M}^*$ be a GBC-model with $\mathcal{M}^* \models \mathrm{On} \to (\mathrm{On})^3_2$, then for any $\gamma \in \mathrm{osp}(\mathcal{M}^*)$,

$$\mathcal{M}^* \models (\forall\alpha\exists\beta)\ [\alpha, \beta] \text{ is } \gamma\text{-dense}.$$

Proof
Case I $\mathcal{M}$ is countable.

We show, by induction on $\gamma \in \mathrm{osp}(\mathcal{M})$,

$$\mathcal{M}^* \models (\forall \text{ proper classes } X \subseteq \mathrm{On})(\exists y \subseteq X)(y \text{ is } \gamma\text{-dense}).$$

For $\gamma = 0$, this is clear because for any cardinal κ and any $\alpha \in X$, one can choose $y \subseteq X$ with $|y| > \kappa$ and $\alpha = \min y$.
To continue the induction, assume the induction hypothesis holds for all $\gamma' < \gamma$.
Let $X \subseteq \mathrm{On}$ be a proper class in $\mathcal{M}^*$.
Then X is a γ-dense class, because for any $F : [X]^3 \to 2$, one chooses a proper class $Y \subseteq X$ that is homogeneous for F (use $\mathcal{M}^* \models \mathrm{On} \to (\mathrm{On})^3_2$) and next, one can choose for any $\gamma' < \gamma$, by using the induction hypothesis, a γ'-dense $y \subseteq Y$ which is then necessarily homogeneous for F.
It therefore suffices to prove that any proper γ-dense class X in $\mathcal{M}^*$ has a γ-dense subset $x \subseteq X$. Let $\mathcal{M}$ be the $\mathcal{L}_{\in}(C)$-structure obtained from $\mathcal{M}^*$ as described in Theorem 7.7.
Choose an elementary extension $\mathcal{N} \succ \mathcal{M}$ with all properties listed in Theorem 7.7, this is possible because of the assumption that $\mathcal{M}$ is countable.

Choose $\delta \in \mathrm{On}^{\mathcal{N}} \setminus \mathrm{On}^{\mathcal{M}}$. Choose $x = X^{\mathcal{N}} \cap \delta$.
Let $f : [x]^3 \to 2$ be a partition in $\mathcal{N}$. Since $\mathcal{N}$ is a conservative extension of $\mathcal{M}$, we find a class function $F : [X]^3 \to 2$ in $\mathcal{M}$ such that for any x_1, x_2, $x_3 \in X^{\mathcal{M}}$, $F(\{x_1, x_2, x_3\}) = f(\{x_1, x_2, x_3\})$. Since $\mathcal{M} \models X$ is γ-dense, we find for any $\gamma' < \gamma$ an $y \subset X$ such that $\mathcal{M} \models y$ is F-homogeneous and γ'-dense.
But then, $\mathcal{N} \models y$ is f-homogeneous and γ'-dense.
We therefore proved that $\mathcal{N} \models x$ is γ-dense. Then $\mathcal{N} \models (\exists x \subseteq X)(x$ is γ-dense$)$ and by elementarity, $\mathcal{M} \models (\exists x \subseteq X)(x$ is γ-dense$)$, what was to be proved.

Case II General $\mathcal{M}$.

We reduce this to the first case using the Downward Löwenheim-Skolem Theorem, in the following way.
Let $\gamma \in \mathrm{osp}(\mathcal{M}^*)$. Choose a countable elementary submodel $\mathcal{N}^* \prec \mathcal{M}^*$ of $\mathcal{M}^*$ that contains γ. By Lemma 8.2, we find $\gamma \in \mathrm{osp}(\mathcal{N}^*)$ and then by the previous case $\mathcal{N}^* \models (\forall\alpha\exists\beta)[\alpha, \beta]$ is γ-dense.
It follows that $\mathcal{M}^* \models (\forall\alpha\exists\beta)[\alpha, \beta]$ is γ-dense. □

Since every meta-theoretic natural number is (after identification of $\mathrm{osp}(\mathcal{M})$ with its ordinal order type) trivially contained in $\mathrm{osp}(\mathcal{M})$, the previous theorem leads to:

Corollary 8.1
For any meta-theoretic $n < \omega$,

$$\mathsf{GBC} + \mathrm{On} \to (\mathrm{On})^3_2 \vdash (\forall\alpha\exists\beta)[\alpha, \beta] \text{ is } n\text{-dense.}$$

This is a scheme that can not be formalised in GBC by a formula of the kind

$$(\forall n < \omega)(\forall\alpha\exists\beta)[\alpha, \beta] \text{ is } n\text{-dense,}$$

because this second statement might talk about non-standard natural numbers in certain GBC-models. It turns out that this sentence even leads to a natural combinatorial incompleteness result for the set theory $\mathsf{GBC} + \mathrm{On} \to (\mathrm{On})^3_2$.

8.2 An incompleteness result

Mc Aloon and Ressayre derive in McAloon and Ressayre (1981) the following transition from provability to unprovability.

Theorem 8.2

- For any meta-theoretic $n < \omega$,

$$\mathsf{GBC} + \mathrm{On} \to (\mathrm{On})^3_2 \vdash (\forall\alpha\exists\beta)[\alpha, \beta] \text{ is } n\text{-dense.}$$

- In contrast,

$$\mathsf{GBC} + \mathrm{On} \to (\mathrm{On})^3_2 \nvdash (\forall n < \omega)(\exists\beta)[0, \beta] \text{ is } n\text{-dense.}$$

- In fact,

$$\mathsf{ZFC} + (\forall n < \omega)(\exists\beta)[0, \beta] \text{ is } n\text{-dense} \vdash \mathrm{Con}(\mathsf{GBC} + \mathrm{On} \to (\mathrm{On})^3_2).$$

To conclude, we guide the reader through the proof of the unprovability part of this theorem (the provability part is exactly Corollary 8.1).

One can take the PA $\Leftrightarrow$ ZFC parallelism one step further by also transferring the Kirby-Paris concept of *strong cut* to models of set theory.

Definition 8.7
Let $\mathcal{M}$ be a ZFC-model.
Let I be a transitive subset of the total order $(\mathrm{On}^{\mathcal{M}}, \in^{\mathcal{M}})$.
Define $V_I = \{x \in M : \mathrm{Rank}^{\mathcal{M}}(x) \in I\}$ and let $\mathcal{M}_I$ be the structure of class theory

$$(V_I, \{V_I \cap x : x \in M\}, \in).$$

We say that I is a *strong cut* of $\mathcal{M}$ if each of the following holds

- $\mathcal{M}_I \models \mathrm{On} \to (\mathrm{On})^3_2$.
- For each $x \in M$, if the set $x_\in$ is cofinal in I, then $\mathcal{M} \models |x| > \beth_{\min x}$.

One can check that if I is a strong cut of a countable ZFC-model $\mathcal{M}$, the structure $\mathcal{M}_I$ satisfies the theory $\mathsf{GBC} + \mathrm{On} \to (\mathrm{On})^3_2$, we refer to McAloon and Ressayre (1981) and Kirby and Paris (1977) for the full proof.

Theorem 8.3
If $\mathcal{M} \models \mathsf{ZFC}$ is countable and I is a strong cut of $\mathcal{M}$, then

$$\mathcal{M}_I \models \mathsf{GBC} + \mathrm{On} \to (\mathrm{On})^3_2.$$

Also the following definition is in complete analogy with the corresponding definition for PA-models.

Definition 8.8
Let $\mathcal{M}$ be a countable model of ZFC.
Let Y be the function in $\mathcal{M}$ defined by $Y(\alpha, \beta) = \max\{\gamma \leq \beta : [\alpha, \beta] \text{ is } \gamma\text{-dense}\}$.

We will say that two ordinals α, β are separated by an ordinal γ if $\alpha \leq \gamma < \beta$.
We will say that two ordinals α, β are separated by a strong cut I if $\alpha \in I < \beta$ (here, $I < \beta$ is short for $(\forall \gamma \in I)(\gamma < \beta)$).

Lemma 8.3
If x is a γ-dense set and z a set of ordinals with $|z| < \min x$, then for any $\gamma' < \gamma$, there is a γ'-dense subset of x such that no two elements of y are separated by an element of z.

Proof
Let $\gamma' < \gamma$. Consider $f : [x]^3 \to 2$ given by:

$$(\forall s \in [x]^3)\ \ (f(s) = 1 \iff (\exists \alpha, \beta \in s)(\exists \theta \in z)(\alpha \leq \theta < \beta))\,.$$

Because x is γ-dense, there is an f-homogeneous γ'-dense subset $y \subseteq x$. It suffices to prove that $f(s) = 0$ for every $s \in [y]^3$. Else, for any three elements of y, there would be an element of z separating two of them. This implies that $|y| \leq |z|$, but then

$$\min y \leq |y| \leq |z| < \min x \leq \min y.$$

This is contradictory. □

Lemma 8.4
Let $\mathcal{M}$ be a ZFC-model and let I be a strong cut of $\mathcal{M}$ containing all standard ordinals of $\mathcal{M}$. Let $y \in V_I$ and γ a standard ordinal of $\mathcal{M}$.
If $\mathcal{M}_I \models y$ is γ-dense, then $\mathcal{M} \models y$ is γ-dense.

Proof
The result follows by induction on the standard ordinals γ of $\mathcal{M}$, noting that any $\mathcal{M}$-function $f : [y]^3 \to \{0,1\}$ is contained in V_I, when $y \in V_I$. □

We can now prove that Y is an indicator for strong cuts in the sense of Kirby and Paris.

Theorem 8.4
Let $\mathcal{M}$ be a countable model of ZFC.
For any pair of non-standard ordinals $\alpha < \beta$ in $\mathcal{M}$:

$$Y(\alpha, \beta) \text{ is non-standard}$$
$$\Updownarrow$$
$$\alpha \text{ and } \beta \text{ can be separated be a strong cut.}$$

Proof
$\boxed{\Leftarrow}$ Suppose $\alpha < \beta$ are non-standard ordinals separated by a strong cut I. We aim for contradiction by assuming that $Y(\alpha, \beta)$ is a standard ordinal of $\mathcal{M}$. Note that I contains all standard ordinals of $\mathcal{M}$, hence $Y(\alpha, \beta)$ is also a standard ordinal of $\mathcal{M}_I$ (Lemma 8.2). By Theorems 8.3 and 8.1, there is $\gamma \in V_I$ such that

$$\mathcal{M}_I \models [\alpha, \gamma] \text{ is } (Y(\alpha, \beta) + 1)\text{-dense.}$$

By Lemma 8.4, $\mathcal{M} \models [\alpha, \gamma]$ is $(Y(\alpha, \beta) + 1)$-dense.
Since $I < \beta$, we have $[\alpha, \gamma] \subseteq [\alpha, \beta]$ and hence

$$\mathcal{M} \models [\alpha, \beta] \text{ is } (Y(\alpha, \beta) + 1)\text{-dense,}$$

but this is in contradiction with the definition of $Y(\alpha, \beta)$.

$\boxed{\Rightarrow}$ Let $\alpha < \beta$ non-standard ordinals in $\mathcal{M}$, suppose $Y(\alpha, \beta)$ is non-standard. This means that there is an infinite decreasing sequence $(\gamma_n)_{n<\omega}$ of $\mathcal{M}$-ordinals with $\gamma_0 < Y(\alpha, \beta)$.
Let $(f_n)_{n<\omega}$ be an enumeration of all partitions $f : [\alpha, \beta]^3 \to \{0,1\}$ in $\mathcal{M}$.
Let $P = \{x \in M : \mathcal{M} \models x \subseteq \mathrm{On}\}$ and let $(z_n)_n$ be a (meta-theoretic) sequence with elements in P such that each element of P occurs infinitely many times in $(z_n)_n$.
We recursively construct a decreasing sequence $(x_n)_{n<\omega}$ of subsets of $[\alpha, \beta]$ that are f_n-homogeneous and γ_{2n}-dense, in the following way.
Let $x_0 = [\alpha, \beta]$. Suppose x_n is already defined and γ_{2n}-dense. Let $x_{n+1}^{(1)}$ be a γ_{2n+1}-dense and f_{n+1}-homogeneous subset of x_n.
If $|z_{n+1}| \geq \min x_{n+1}^{(1)}$, we set $x_{n+1} = x_{n+1}^{(1)}$. If $|z_{n+1}| < \min x_{n+1}^{(1)}$, we choose, using Lemma 8.3, $x_{n+1} \subseteq x_{n+1}^{(1)}$ that is γ_{2n+2}-dense such that no two elements of x_{n+1} are separated by an element of z_{n+1}. It follows that:

- x_n is γ_{2n}-dense

- x_n is f_n-homogeneous
- If $|z_n| < \min x_n$, then no two elements of x_n are separated by an element of z_n.

Define $\alpha_n = \min x_n$ and $\beta_n = \sup x_n$.
We check that $I := \bigcup_n (\alpha_n)_\in$ is a strong cut separating α and β.
First, we claim that for any $l < \omega$, there is $k > l$ such that $\alpha_l < \alpha_k$. If this would not be the case, we would have $(\forall k \geq l)\ \alpha_l = \alpha_k$.
Consider then the partition $f : [a, b]^3 \to 2$:

$$f(\{x_1, x_2, x_3\}) = 1 \iff \{x_1, x_2, x_3\} \subseteq [\alpha_l + 1, \beta_l].$$

For k large enough, we have that x_k is F-homogeneous, but since $\min x_k = \alpha_l$, we certainly find three different elements of x_l that are in $[\alpha_l + 1, \beta_l]$. This implies that all elements of x_k are in $[\alpha_l + 1, \beta_l]$, which is not true.
Next, we claim that in fact

$$I = \bigcup_n (\alpha_n)_\in = \bigcap_n (\beta_n)_\in.$$

If not, then we can choose $\xi \in \bigcap_n (\beta_n)_\in \setminus \bigcup_n (\alpha_n)_\in$.
Consider the partition $f : [a, b]^3 \to 2$:

$$f(\{x_1, x_2, x_3\}) = 1 \iff \{x_1, x_2, x_3\} \subseteq [a, \xi[\vee \{x_1, x_2, x_3\} \subseteq [\xi, b].$$

For k large enough, we have that x_k is f-homogeneous, but this implies that either $x_k \subseteq [a, \xi[$ or $x_k \subseteq [\xi, b[$, so $\beta_k = \sup x_k \leq \xi$ or $\alpha_k = \min x_k \geq \xi$. Since $\xi \in \bigcap_n (\beta_n)_\in \setminus \bigcup_n (\alpha_n)_\in$, this is only possible when $\alpha_k = \xi$. This implies therefore that $\alpha_k = \xi$ for all k large enough, which we found to be impossible.
Let x be cofinal in I. If $|x| > I$ or $|x| = I$, then there is n such that $\min x \leq \alpha_n$ and $\beta_n \leq |x|$. Then:

$$\beth_{\min x} \leq \beth_{\alpha_n} < |x_n| \leq \beta_n \leq |x|.$$

Else, there is n such that $|x| < \alpha_n$ and $x = z_n$.
By construction, it follows that in this case no two elements of x_n are separated by an element of x, but then x can not be cofinal in x_n.
By construction, we have that $\mathcal{M}_I \models \mathrm{On} \to (\mathrm{On})^3_2$. If $F : [V_I]^3 \to \{0, 1\}$ is a class partition in $\mathcal{M}_I$, then F coincides with certain $f : [\alpha, \beta]^3 \to \{0, 1\}$ on $[\alpha, \beta]^3 \cap V_I$. However, for certain $n < \omega$, x_n is f-homogeneous. Then $x_n \cap I$ is a F-homogeneous proper class of $\mathcal{M}_I$. □

Proof Sketch of Theorem 8.2
Suppose $\mathcal{M} \models \mathsf{ZFC} + (\forall n < \omega)(\exists \beta [0, \beta]$ is n-dense$)$.
We continue working *inside* $\mathcal{M}$.
Let T be a suitably selected finite subtheory of ZFC.
By reflection, $T + (\forall n < \omega)(\exists \beta [0, \beta]$ is n-dense$)$ is satisfied in certain V_α.
By compactness,

$$T + (\forall n < \omega)(\exists \beta [0, \beta] \text{ is } n\text{-dense}) + \{c > \underbrace{1 + \ldots + 1}_{k \text{ terms}} : k < \omega\} + c < \omega$$

is satisfiable.
By the Downward Löwenheim-Skolem Theorem, we can choose a countable model $\mathcal{N} = (N, \in^{\mathcal{N}}, c^{\mathcal{N}})$ of the last theory.
Let $\rho = c^{\mathcal{N}}$. Then $\mathcal{N} \models (\exists\beta)([0,\beta]$ is ρ-dense). By Theorem 8.4 and Theorem 8.3 there is a strong cut I of $\mathcal{N}$ such that $\mathcal{N}_I \models \mathsf{GBC}+\mathrm{On} \to (\mathrm{On})^3_2$. By executing the previous inside $\mathcal{M}$, we have constructed in $\mathcal{M}$ a model of $\mathsf{GBC}+\mathrm{On} \to (\mathrm{On})^3_2$, it follows that $\mathcal{M} \models \mathrm{Con}(\mathsf{GBC} + \mathrm{On} \to (\mathrm{On})^3_2)$. □

References and Remarks

The article McAloon and Ressayre (1981) offers various further ideas for translating independence principles for PA to set theories. This still seems a fruitful challenge and it is unclear to what extent it has been taken up already. Also a further study on exactly how this set theoretical versions of combinatorial PA independence results relate to large cardinals could be very interesting.

Nederlandse samenvatting

In deze masterthesis wordt toegewerkt naar een nieuw onvolledigheidsresultaat dat aansluit bij de hoofdresultaten uit Weiermann (2003). Dit onvolledigheidsresultaat wordt in deze thesis geformuleerd voor de zwakke verzamelingenleer Sim, die door Stephen Simpson werd ingevoerd in Simpson (1982).

Hoofdstuk 1 behandelt enkele systemen van zwakke verzamelingenleer en voert ook de theorie Sim in. Vervolgens bespreken we in Hoofdstuk 2 de eigenschappen van een specifieke klasse van gewortelde bomen (we verkozen de naam Γ-bomen) en definiëren een relatie op de verzameling van dergelijke bomen. We gaan vervolgens na, gebruik makend van inductie op natuurlijke getallen (en geen andere infinitaire principes), dat deze relatie een totale orderelatie vormt. In Hoofdstuk 3 bestuderen we de asymptotiek van de bijhorende telfuncties van dergelijke families van bomen, aan de hand van de analytische eigenschappen van hun genererende functies. We voeren hierbij de benodigde technieken uit de analytische combinatoriek in. We maken hierbij onder andere gebruik van de preparatiestelling van Weierstraß, die we ook bewijzen.

Er blijken ook meerdere interessante grafentheoretische links te zijn met deze Γ-bomen en dit wordt besproken in Hoofdstuk 4. In Hoofdstuk 5, wordt de link gelegd tussen Γ-bomen en het in datzelfde hoofdstuk ingevoerde verzameling-theoretische ordinaalgetal Γ_0. We bestuderen verder de eigenschappen van dit grote aftelbare ordinaal en met behulp van verzamelingenleer kunnen we nu ook aantonen dat de orde op Γ-bomen een welorde vormt. Het aangekondigde onvolledigheidsresultaat wordt ook behandeld in Hoofdstuk 5. Hierbij passen we de argumenten voor PA-onafhankelijkheidsresultaten uit Weiermann (2003) aan voor de verzamelingentheorie Sim.

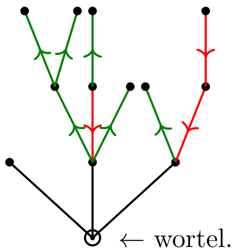
Het toepassen van dezelfde intuïtie op resultaten over einduitbreidingen van PA-modellen leidt tot het materiaal dat besproken wordt in Hoofdstuk 6 en Hoofdstuk 7. Hierbij komt het modeltheoretische begrip van zogenaamde "types" goed van pas. We bewijzen een krachtige stelling die toelaat om modellen te creëren waarin bepaalde types vermeden worden. We maken hiervan gebruik om elementaire einduitbreidingen te construeren. We geven in Hoofdstuk 7 onder andere een volledig bewijs van een resultaat over ranguitbreidingen van bepaalde modellen van de klassentheorie GBC dat in het artikel McAloon and Ressayre (1981) enkel geschetst werd. Hoofdstuk 8 geeft een overzicht van enkele resultaten uit datzelfde artikel die in verband staan met de methode van Kirby en Paris voor het bewijzen van onvolledigheidsresultaten met behulp van zogenaamde indicatoren. Onder andere "α-dichte verzamelingen" worden ingevoerd en besproken.

Communicatie naar een breed publiek

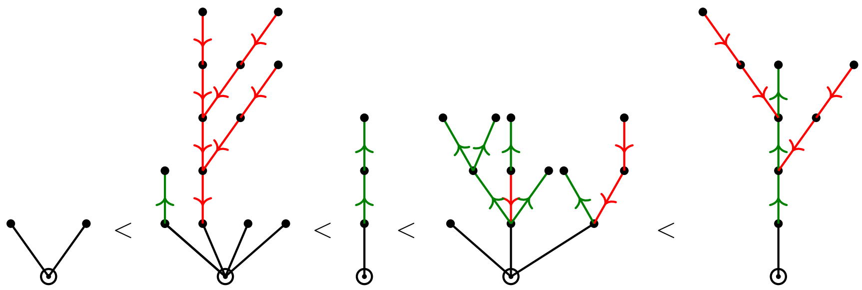
Twee woorden die erg vaak met wiskunde worden geassocieerd zijn “tellen” en “bewijs”. Laat ons over beide begrippen iets meer vertellen.

Tellen met bomen

Een belangrijk begrip in deze thesis is dat van de Γ-boom.
Een Γ-boom is niets meer dan een figuur die er als volgt uitziet:

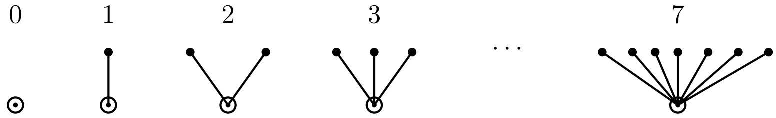


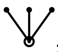
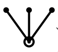
In deze thesis wordt een orderelatie gedefinieerd op de verzameling der Γ-bomen. Dat betekent dat we voor elke twee verschillende Γ-bomen vastleggen welk van beide de grootste is.
Om te weten te komen hoe deze ordening precies gedefinieerd is, verwijzen we graag door naar Definitie 2.11, maar we kunnen hier wel al enkele voorbeelden beschouwen.



Deze ordening is interessant omdat deze bomen zich zo erg goed lenen tot het tellen van de objecten in andere geordende verzamelingen.

Voorbeeld 1: Elementen van een eindige verzameling.
Beschouwen we even de verzameling $A = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Dan kunnen we de elementen van A als volgt "tellen" met Γ-bomen:

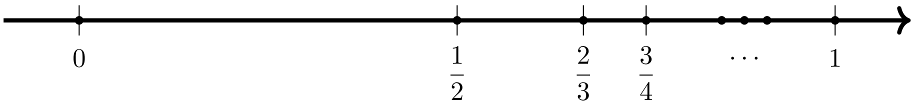


Het lijkt misschien meer voor de hand liggend om symbolen als 1 en 3 te gebruiken dan ⊙ en , maar we kunnen hierbij de bedenking maken dat het symbool 1 niets meer is dan een gestileerde vorm van één geturfd streepje (en dat de romeinse III, bestaande uit 3 geturfde streepjes, wel heel wat gelijkenis vertoont met ).

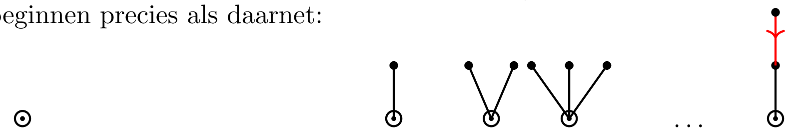
Voorbeeld 2: $A = \left\{1 - \frac{1}{n} : n \in \mathbb{N}_{>0}\right\} \cup \{1\}$.
Een voordeel van onze boom-notatie is dat we hiermee ook bepaalde oneindige verzamelingen van klein naar groot kunnen tellen.
Beschouw even bovenstaande

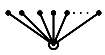
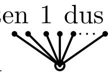
$$A = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\} \cup \{1\}.$$

Getekend op de reële rechte ziet deze er als volgt uit:



We beginnen precies als daarnet:

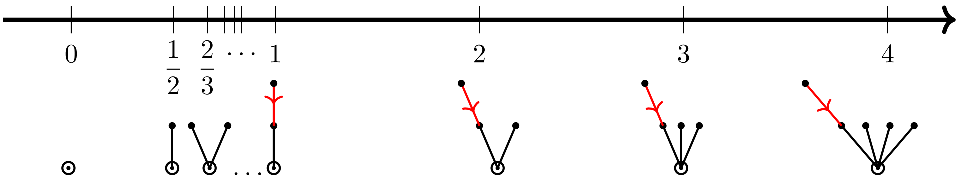


We gebruiken dus alle bomen van de vorm  om de elementen van $\{1 - \frac{1}{n} : n \in \mathbb{N}_{>0}\}$ te tellen. Daarna zijn we nog niet klaar omdat we 1 vergaten te tellen. Onthoud dat we van klein naar groot tellen! We moeten 1 dus tellen met de kleinste boom die groter is dan alle bomen van de vorm . Het interessante is dat de ordening die we definiëren op Γ-bomen precies zo'n element heeft, namelijk de boom .
Hiermee is het telsysteem met Γ-bomen krachtiger dan het systeem $\{0, 1, 2, \ldots\}$, omdat we met dit laatste systeem bovenstaande verzameling A nooit van klein naar groot zouden kunnen tellen (immers, met welk getal zouden we $1 \in A$ moeten tellen?).

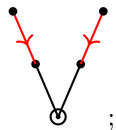
Met Γ-bomen kunnen we zelfs zo ver tellen dat het al heel moeilijk voor te stellen is hoe een verzameling met bijvoorbeeld een  -de element er zou moeten uitzien.

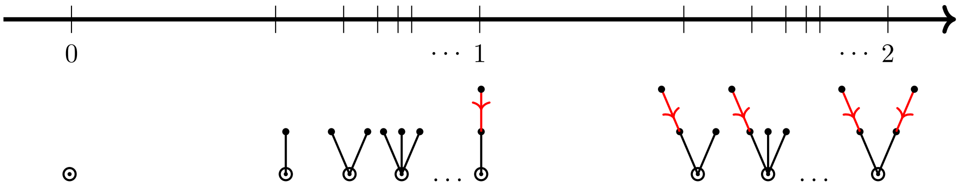
Nog enkele voorbeelden:

Voorbeeld 3: $A = \left\{ 1 - \frac{1}{n} : n \in \mathbb{N}_{>0} \right\} \cup \{1, 2, 3, 4\}$.



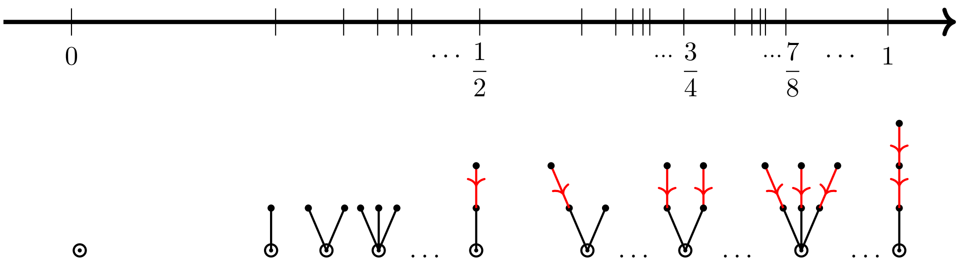
Voorbeeld 4: $A = \left\{ 1 - \frac{1}{n} : n \in \mathbb{N}_{>0} \right\} \cup \left\{ 2 - \frac{1}{n} : n \in \mathbb{N}_{>0} \right\} \cup \{2\}$.

We tellen hierbij twee keer tot , dus het is logisch dat we uitkomen bij  ; toch?



Om tot nog grotere bomen te kunnen tellen, moeten we wel al iets ingewikkeldere verzamelingen beschouwen.

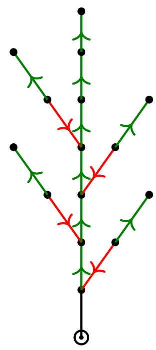
Voorbeeld 5: $A = \left\{ 1 - 2^{-k+1} + \frac{1 - \frac{1}{n}}{2^k} : k, n \in \mathbb{N}_{>0} \right\} \cup \{1\}$.



Deze A ziet er uit als  kopieën van $\{1 - \frac{1}{n} : n \in \mathbb{N}_{>0}\}$ achter elkaar geplakt: voor elke $k \in \mathbb{N}_{>0}$ bevat ze een nieuwe kopie van $\{1 - \frac{1}{n} : n \in \mathbb{N}_{>0}\}$. Zo geraken we net tot . Tot dusverre hebben we, hoever we ook al geteld hebben, nog nooit tot aan een Γ-boom kunnen tellen waarin opwaarts gerichte pijlen voorkomen.

Hoe tellen we dan tot aan ? Wel,  is de kleinste Γ-boom groter dan alle bomen van de vorm , , , …, , dus als we bij het tellen op een punt komen waarin deze allemaal geteld zijn, en er is nog een kleinste niet geteld object, dan gebruiken we daarvoor . Om dit te verwezenlijken, moeten we wel een heel ingewikkelde geordende verzameling aan het tellen zijn. Er zijn voorbeelden te vinden van zulke verzamelingen onder de deelverzamelingen van de rationale getallen, maar het vergt al heel wat moeite om zo'n verzameling neer te schrijven.

Dit alles in acht genomen, is het duidelijk dat een geordende verzameling die ons bij het tellen tot een boom als



leidt wel heel ingewikkeld moet zijn. Daarom is het best verrassend dat we dergelijke overzichtelijke boom kunnen gebruiken om deze te begrijpen.

Bewijzen

Wat wiskunde het meest onderscheidt van alle (andere) wetenschappen is de centrale status die aan het begrip "bewijs" wordt toegewezen.

De axioma's van een theorie bepalen, samen met de deductieregels van het bewijssysteem, wat er bewijsbaar is in die theorie. Een uitspraak zoals

$$1 + 1 = 3$$

zouden we idealiter niet bewijsbaar wensen, maar de negatie van deze uitspraak,

$$1 + 1 \neq 3,$$

zouden we wel graag kunnen bewijzen. Wanneer een uitspraak, noch zijn negatie, bewijsbaar zijn in een bepaalde theorie, noemen we deze uitspraak onafhankelijk van deze theorie. De onbewijsbare zinnen kan men dus zien als hiaten van de theorie.
In deze masterthesis onderzoeken we een zin φ_r, die afhangt van de rationale parameter $r \in \mathbb{Q}$. We bewijzen dat deze zin voor voldoende grote rationale getallen onafhankelijk wordt van een bepaalde verzamelingentheorie. Is r echter kleiner, dan is de zin gewoon bewijsbaar in de bewuste verzamelingentheorie. In deze thesis bepalen we de bewuste grenswaarde.
Waarover kan zo'n zin φ_r dan gaan? Wel, φ_r is een uitspraak over de eigenschappen van de eerder besproken Γ-bomen. Omdat Γ-bomen zulke eenvoudige objecten zijn vraagt het niet veel van een theorie om over deze eigenschappen te kunnen spreken. Echter, precies omdat deze in staat zijn dergelijke gecompliceerde geordende verzamelingen te tellen, blijkt het wel veel te vragen van een theorie om bepaalde eigenschappen van deze bomen te bewijzen.

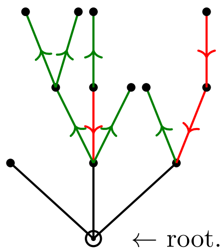
Communication to a wider audience

Two words that are very often associated with mathematics are “counting” and “proof”. Let us explore these notions further.

Counting by trees

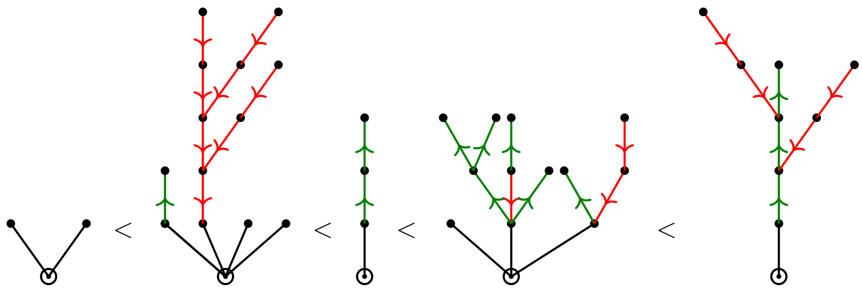
An important notion in this thesis is the notion of a Γ-tree.

A Γ-tree is simply a figure that looks like this:



In this thesis an order relation is defined on the set of the Γ-trees. This means that for any two different Γ-trees we state which of both is the largest.

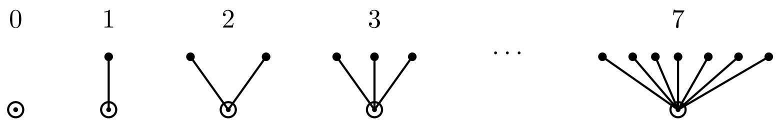
For more information on how this order relation is precisely defined, we refer to Definition 2.11. Let’s consider here the following examples.

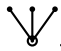
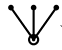


This order is interesting because these trees can be conveniently used for counting the objects of certain ordered sets.

Example 1: Elements of a finite set.

Consider the set $A = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Then we can “count” the elements of A with Γ-trees as follows:

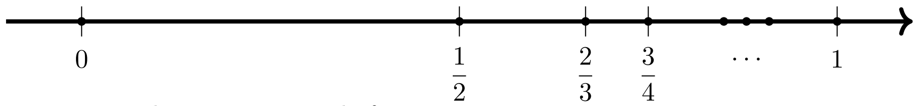


It may seem more obvious to use symbols 1 and 3 instead of ⊙ and , but note that the symbol 1 is simply a stylized form of one tally mark (and that the roman III, or three tally marks, shows substantial resemblance with ).

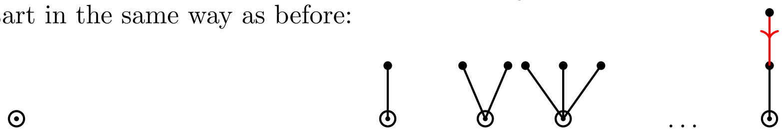
Example 2: $A = \{1 - \frac{1}{n} : n \in \mathbb{N}_{>0}\} \cup \{1\}$.
An advantage of the tree-notation is that it can be used to count certain infinite sets from small to large.
Consider the set

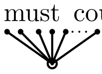
$$A = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\} \cup \{1\}$$

mentioned above. Indicated on the real axis it looks as follows:



We start in the same way as before:

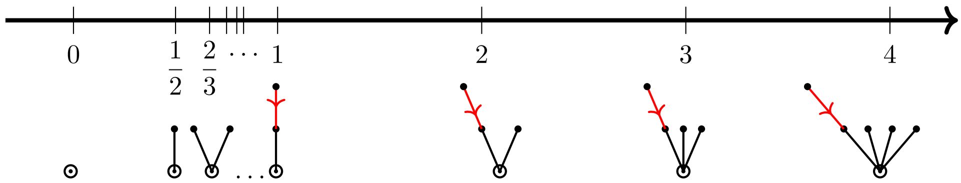


So we use all trees of the form  to count the elements of $\{1 - \frac{1}{n} : n \in \mathbb{N}_{>0}\}$. After this we are not done yet, because we forgot to count 1. Remember that we count from small to large! This means we must count 1 with the smallest tree that is larger than all trees that look like . Interestingly, the ordering that we define on Γ-trees has exactly one such an element, namely the tree .
This implies that the Γ-trees counting system is more powerful than the system $\{0, 1, 2, \ldots\}$, because we could never count the set A mentioned above from small to large with the latter system (what number would we use to count $1 \in A$?).

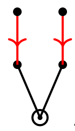
Γ-trees even allow us to count so far that it is already very difficult to imagine how a set with for instance a -th element would have to look like.

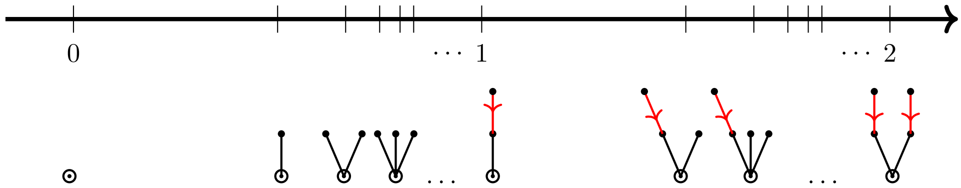
Some more examples:

Example 3: $A = \left\{1 - \frac{1}{n} : n \in \mathbb{N}_{>0}\right\} \cup \{1, 2, 3, 4\}$.



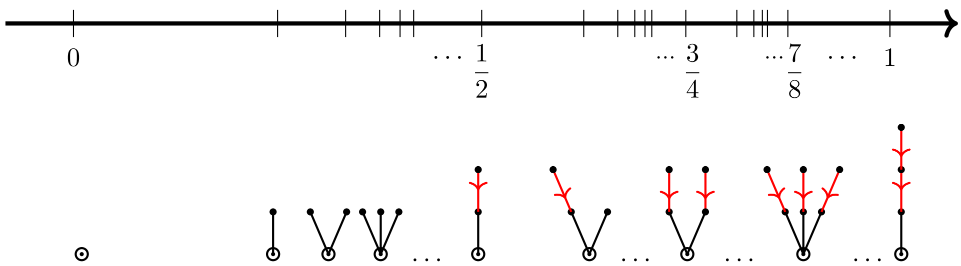
Example 4: $A = \left\{1 - \frac{1}{n} : n \in \mathbb{N}_{>0}\right\} \cup \left\{2 - \frac{1}{n} : n \in \mathbb{N}_{>0}\right\} \cup \{2\}$.

In this example we count twice to , so it makes sense that we end up at .



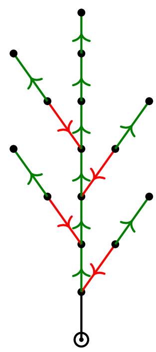
In order to count to even bigger trees, we have to consider some more complicated sets.

Example 5: $A = \left\{1 - 2^{-k+1} + \frac{1 - \frac{1}{n}}{2^k} : k, n \in \mathbb{N}_{>0}\right\} \cup \{1\}$.



This A looks like  copies of $\{1-\frac{1}{n} : n \in \mathbb{N}_{>0}\}$ pasted after each other: for each $k \in \mathbb{N}_{>0}$ it contains a new copy of $\{1-\frac{1}{n} : n \in \mathbb{N}_{>0}\}$. This takes us precisely up to . Even though we have counted already quite far, we have so far never been able to count up to a Γ-tree that contains upward-directed arrows.

So how do we count up to ? Well,  is the smallest Γ-tree larger than all trees of the form , , , ..., , So when we are counting and we arrive at a stage where we have counted all these trees, and there remains a smallest object that is not yet counted, then for this object we will use . To accomplish this, we must be counting very complicated ordered sets. Examples of such sets can be found in the subsets of the rational numbers, but it takes much more effort to write down such a set.
All this considered, it is clear that an ordered set that leads us whilst counting to a tree like



must be very complicated. For that reason it is quite surprising that we can use such a compact tree to understand this set.

Proofs
The main aspect that distinguishes mathematics from all (other) sciences is the central status that it assigns to the notion of a "proof". The axioms of a theory determine, together with the deduction rules of the proof system, what is provable in that theory. Ideally, we would like a sentence such as

$$1 + 1 = 3$$

to be non-provable, but we would like to be able to prove the negation of this sentence,

$$1 + 1 \neq 3.$$

If a sentence, nor its negation, is provable in a certain theory, then we call this sentence independent of this theory. The unprovable statements can therefore be seen as gaps of the theory.

In this master's thesis we investigate a sentence φ_r, that depends on the rational parameter $r \in \mathbb{Q}$. We prove that this sentence becomes independent of a certain set theory for sufficiently large rational numbers. But if r is smaller, then the sentence is provable in this particular set theory. In this thesis we determine the specific limit value for provability.

What can be the content of such a sentence φ_r? Well, φ_r is a statement about the properties of the Γ-trees that were mentioned earlier. Because Γ-trees are such simple objects, it does not take much of a theory to formalise these properties. Precisely because they are capable to count such complicated ordered sets, it does demand a lot of a theory to prove some of the properties of these trees.

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