On Quine's Set theory

Roland Hinnion's Ph.D. thesis
translated by Thomas Forster

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Translator's Preface

Roland Hinnion was the first of Boffa's *Enéfiste protégés* to complete. Other members of the Séminaire NF that worked under Boffa's guidance—-if not his formal direction—were Marcel Crabbé, André Pétry and Thomas Forster, all of whom went on to complete Ph.D.s on NF. This work of Hinnion's has languished unregarded for a long time. Indeed I don't think it has ever been read by anyone other than members of the Seminaire NF and the handful of people (such as Randall Holmes) to whom copies were given by those members. In it Hinnion explains how to use relational types of well-founded extensional relations to obtain models for theories of well-founded sets: to the best of my knowledge this important idea originates with him. Variants of it appear throughout the literature on set theory with antifoundation axioms and I suspect that Hinnion's work is not generally given its due. One reason for this is that its author no longer works on NF. Another reason is the gradual deterioration of language skills among English-speaking mathematicians: the proximate cause of my embarking on this translation was the desire of my student Zachiri McKenzie to familiarise himself with Hinnion's ideas, and McKenzie has no French. I am grateful to him for reading through my translation to check for errors and infelicities.

The original has no footnotes. This leaves the footnote as the obvious place for the translator to place helpful commentary, and I have put it to that use. Minor and brief observations of the translator not meriting a footnote are occasionally to be found inserted in line in the text enclosed in square brackets.

The translation is fairly free: one must not allow oneself to be spooked by the possibility of the original being unfaithful to the translation, and my primary aim is simply to make Hinnion's ideas accessible to monoglot anglophones. I have generally tried to stick to the author's original notation, but there are some notations that are now so archaic that use of them in a third millenium document would be perverse—tho' for the benefit of those who wish to keep track of changing notation I shall at least record here the changes I have made. I write '$|x|$' instead of the Rosser-ism '$Nc(x)$' for the cardinal of x; 'ι"x' for '$USC(x)$'; '$\mathcal{P}(x)$' for '$SC(x)$' and 'V_α' instead of 'R_α'.... Reluctantly I have also changed 'Λ' to '$\emptyset$' throughout to conform with deplorable modern practice. (It is ironical that the notation 'V' for the universal set, which arose from the notation 'Λ' for the empty set by turning it upside-down, is still in use!) In addition, LaTeX has enforced certain numbering systems that I am powerless to overrule. On the plus side we have with LaTeX a wide variety of felicitous notations that were not available to Hinnion and his typewriter in 1974, (I write '$\blacksquare$' to signal the end of a proof, for example) and some notations have changed slightly but significantly since Hinnion wrote this work. For Hinnion a *relation bien fondée extensionnelle* is a binary relation that is well-founded and extensional. In Cambridge my Ph.D. students and I have been using the expression 'BFEXT' to denote binary structures $\langle A, R\rangle$ where R is a well-founded extensional relation *with a "top" element*. Since the objects of interest in this context are these structures (with the knobs on, almost literally) rather than the slightly more general binary relations that Hinnion denotes with the expression it seems sensible to reserve the nice snappy notation for them, the objects of interest. I have also changed Hinnion's 'ω_R' notation for the "top" element of the domain of a well-founded extensional relation to '$\mathbb{1}_R$', to conform to current practice here in Cambridge. Hinnion used Roman letters for variables ranging over relational types of BFEXTs; I have used lower-case $\mathfrak{fraktur}$ font for variables ranging over those relational types. I have tried to use upper-case $\mathfrak{FRAKTUR}$ font for variables ranging over structures. Part of the attraction of this notation is the scope it gives for using the corresponding uppercase Roman letter to denote the carrier set.

There are occasional typing mistakes in the original. I have corrected those I have found—and usually without comment: this is not a critical edition.

Various works alluded to in the footnotes have been added to the bibliography. The items in the original bibliography appear here in their original order; the new items have been appended on the end.

I am grateful to Roland Hinnion for clarification on some points of detail, and to Randall Holmes and Zachiri MacKenzie for reading draughts of this translation.

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Introduction

NF (abbreviation of "New Foundations") is the Set Theory invented by Quine in [3] and developed by Rosser in [4]. It is finitely axiomatisable [5]. It is usually axiomatised by means of the axiom of extensionality and a scheme of comprehension for stratified formulæ. A formula is stratified if one can decorate the variables in the formula in such a way that the result is a formula of the simple theory of types. NF asserts that if ϕ is stratified then the set $\{x : \phi\}$ exists.[1]

One can in NF define such usual notions as union, intersection, cartesian product (see the list of definitions p. 11). However one can also define some sets that have no equivalent in traditional set theories such as ZF and Z (these theories are defined in chapters 4 and 5. Notations there are from [6] pp 507–9.)

In NF we have the existence of a universal set (the set of all sets) and a set NO of all ordinals (an ordinal is an isomorphism class of well-orderings). Because of the stratification conditions in NF these sets do not give rise to contradiction.

One can show that the set $\mathbb{N}$ of natural numbers in NF (defined as the set of equipollence classes) satisfies the axioms of Peano arithmetic (not forgetting that induction holds only for stratified formulæ. Remember that verifying the Peano axiom that says that "+1" is injective relies on the axiom of infinity in the form "$V \notin Fin$" (Fin being the set of finite sets), which was established by Specker [7]. Specker's result is actually that V cannot be well-ordered.[2] The axiom of choice is thus false [*sic*] in NF.

The axiom of foundation is clearly also false [*sic*] in NF, because $V \in V$: there are cyclic membership relations in NF. Accordingly it is natural to ask what sort of relations can be represented by "$\in$" in NF. An early result of Boffa's in this direction (see chapter 1) shows that—at least if NF is consistent—then every finite extensional structure has an end-extension that is a model of NF. We obtain as a corollary that every [binary] sstructure can be embedded in some model of NF. Boffa has obtained results analogous to this for fragments

[1] The set abstract is allowed to contain parameters.

[2] Specker [7] published a refutation of AC in NF. Since AC can be proved to hold for finite sets by a stratified induction, one obtains the axiom of infinity as a corollary. There is a quite separate (more digestible) proof of the Axiom of Infinity—using the same device of cardinal trees—which Specker knew but never published, the published result being stronger. The more digestible proof has been rediscovered several times.

of ZF-minus-foundation. See [8].

One result in chapter 1 [6] shows in effect that every finite [binary] structure embeds in every model of NF. One corollary of this is that every universal sentence is either provable or refutable in NF. (Analogous results for fragments of ZF-minus-foundation are to be found in [8]).

These results all follow from the proposition—provable in NF—that, for any structure $\langle A, R\rangle$ where A is strongly cantorian, there is a set B such that $\langle A, R\rangle \simeq \langle B, \in\rangle$.

We say a set is **cantorian** if A is the same size as $\iota``A := \{\{t\} : t \in A\}$. By formalising Cantor's Paradox we discover that V is not cantorian; by formalising the Burali-Forti paradox we discover that NO is not cantorian. In contrast we can show that $\mathbb{N}$—the set of naturals—is cantorian. A set A is **strongly cantorian** if there is a function that sends t to $\{t\}$ for all $t \in A$. An interesting axiom, proposed by Rosser, and called by him "the Axiom of Counting" is the assertion that $\mathbb{N}$ is *strongly* cantorian. Henson [2] has shown that the Axiom of Counting is not provable in NF.[3] It remains an open question whether or not the Axiom of Counting is actually refutable in NF.

The importance of this axiom is pointed up by a result of Henson [1] to the effect that if NF + the Axiom of Counting is consistent then so is NF + "The axiom of Counting + every well-ordering of a strongly cantorian set is isomorphic to a von Neumann ordinal". In particular the consistency of NF + the Axiom of Counting implies the consistency of NF + an axiom asserting the existence of the von Neumann ordinal ω. Observe that the von Neumann ω cannot be defined by a stratified formula. (This is proved in [8] p 277.)

One might wonder whether the consistency of NF + the Axiom of Counting implies the consistency of NF + axioms affirming the existence of Z (the set of Zermelo naturals) or V_ω (the set of well-founded hereditarily finite sets). (Work of Boffa [8] shows [4] that neither Z nor V_ω can be defined by stratified formulæ.) We obtain an affirmative answer as a consequence of a result (proposition 35 in chapter 2, p. 27) which generalises a theorem of Henson [1].

Henson's result exploits the [Rieger-Bernays] permutation method and the structure $\langle NO, <\rangle$ where $<$ is the obvious well-order of the set of all ordinals. In chapter 2 we show how to extend[5] $\langle NO, <\rangle$ to a structure $\langle BF, \mathcal{E}\rangle$. The elements of BF are the isomorphism classes of well-founded extensional relations with a unique maximal element.[6] We can find a natural relation $\mathcal{E} \subseteq (BF \times BF)$ such that $\langle BF, \mathcal{E}\rangle$ is a well-founded extensional structure. Henson's proof works just as well if we replace 'NO' by 'BF', and '$<$' by '$\mathcal{E}$', and we get the following result:

[3] This is not quite correct. Orey [**?**] showed that NF + the axiom of Counting $\vdash$ Con(NF).

[4] Hinnion mentions Boffa but the cited article is by Jensen.

[5] The adverb *transitivement* appears in the original. I think the author was alluding to the fact that $\langle BF, \mathcal{E}\rangle$ is an end-extension of $\langle NO, <\rangle$.

[6] This is open to misinterpretation, since well-founded extensional relations need not be transitive. The well-founded extensional relations the author has in mind are those relations R which have in their domain a unique element R^* such that every element in the domain of R has an R-path to R^*. Such a binary relation R looks naturally like the membership relation restricted to a (well-founded) transitive set of the form $TC(\{x\})$.

> *If T is a consistent invariant extension of NF then the theory T + "every well-founded extensional structure $\langle A, R\rangle$ where A is strongly cantorian is isomorphic to a structure $\langle B, \in\rangle$ where B is a transitive set" is consistent.*

The precise definition of the expression 'invariant' is given in [1] [page 70 defn. 1.3]; for the moment note merely that every stratified sentence is invariant and that the Axiom of Counting is invariant. An invariant extension of NF is an extension all of whose axioms are invariant.

The result quoted above shows that we can consistently add to any consistent invariant extension of NF the principle of contraction [Mostowski collapse] for strongly cantorian well-founded extensional structures. In particular, if NF + the Axiom of Counting is consistent, so is NF + the existence of Z, ω [the von Neumann ω] and V_ω.

The chief problem concerning NF is that of its consistency. So far all attempts to construct a model of NF have ended in failure. One partial result in that direction is that of Jensen [9]. Jensen has shown the consistency of NFU in arithmetic, where NFU is the system obtained from NF by restricting the axiom of extensionality to nonempty sets. It is interesting to study the strength of NF relative to its subsystems the better to ascertain in which systems we can (perhaps) construct models of NF. Orey has obtained models of ZF in some (quite strong) extensions of NF: [10] and [11].

In [10] Orey shows the following. Let NF$'$ be the system obtained by adding to NF

1. A scheme of induction for all formulæ over strongly cantorian ordinals; [7]

2. An axiom asserting that if α is a strongly cantorian ordinal then so is ω_α;[8]

3. The scheme E_1: If FC is the class of strongly cantorian ordinals and ϕ is any formula then

   $(\forall\alpha \in FC)(\exists\beta \in FC)(\phi(\alpha, \beta) \to$
   $(\forall x \subset FC)(\exists\alpha \in FC)(\forall a \in x)(\forall b \in FC)[\phi(a, b) \to b < \alpha])$

In this last scheme the variable 'x' ranges over subsets of FC. The scheme aserts that for any such set x and any function from FC to FC the image of x in that function is bounded by an element of FC. Orey's result now states that we can prove the consistency of ZF in NF$'$.

[7] *Explanatory note added by author:* An ordinal α is said to be strongly cantorian if $\{\beta : \beta < \alpha\}$ is a strongly cantorian set. Normally one can do induction over the ordinals only for <u>stratified</u> formulæ. The scheme added by Orey extends this to unstratified formulæ, at the cost of restricting it to strongly cantorian ordinals.

[8] *Explanatory note by author:* for a precise definition of ω_α see page 12. Note that since 0 is strongly cantorian this implies that ω_0 is strongly cantorian which implies the Axiom of Counting. Translator's note: Holmes claims that this is stronger than the Axiom of Counting. Surely if $\alpha \geq \omega$ is strongly cantorian then ω_α exists and is cantorian ...but not obviously strongly cantorian

In [11] Orey introduces the system NF$''$ which is NF + a scheme of induction over FC for all formulæ, Rosser's Axiom of Counting and the scheme E_1. The result now is that Con(NF$''$)$\rightarrow$ Con(ZF).

Orey does this by constructing models of ZF in the ordinals of NF. That is why he needs extra axioms—to make the theory of the ordinals strong enough. This method resembles the construction of L.[9]

The results of chapters 4, 5, 6 and 7 show that it is possible to obtain relative consistency results for fragments of ZF by adding to NF axioms about *cardinals*. Recall that in NF cardinals are equipollence classes.

We can define substructures of $\langle BF, \mathcal{E} \rangle$ corresponding to the various V_α. Thus one can obtain natural models of fragments of ZF.

The structure $\langle BF, \mathcal{E} \rangle$ is interesting also in that enables one to recover the results of Orey (In effect one defines analogues of the L_α of ZF).

The results obtained by use of $\langle BF, \mathcal{E} \rangle$ are the following:

1. In NF, the assumption that $\Phi(\aleph_0)$ is infinite implies the consistency of $Z+$ TC. If α is the cardinal of ι“A, the 2^α is the cardinal of $\mathcal{P}(A)$. If α is not a cardinal of this form then by convention 2^α is the empty set.[10]

   This result shows that NF + “$\Phi(\aleph_0)$ is infinite” proves Con(Z).

   Note that this extension of NF is *stratified* and that NF + Axiom of Counting $\vdash \Phi(\aleph_0)$ is infinite. We know from work of Henson [2] that there are stratified sentences that imply the consistency of Z [when added to NF] and which are weaker than the Axiom of Counting—for example Con(Z)! It remains to find stratified such sentences having a natural meaning in NF. “$\Phi(\aleph_0)$ is infinite” is such a sentence. We do not at this stage know whether or not this sentence is a theorem of NF. It is closely related to a question of Specker's: is the set of infinite cardinals infinite?[11]

2. In NF the existence of an infinite strongly inaccessible cantorian cardinal implies the consistency of ZF. A cardinal is said to be cantorian if it is the cardinal of a cantorian set.[12] [definition of strongly inacessible deleted]

3. In NF + “a union of countably many countable sets is countable” we can construct a model of ZF$\backslash P$ (ZF minus the power set axiom). The additional axiom is a weak form of the axiom of choice. This shows that it could be interesting to investigate the consistency, relative to NF, of other weak forms of choice. At present we know nothing.[13]

4. Attempts to interpret Z in an un-reinforced NF have not had the desired result. In Chapters 6 and 7 we show which fragments of ZF are the

[9] This method was anticipated by Firestone [13] and Takeuti [18].

[10] TC = axiom of transitive closure. The ‘Φ’ notation is from Specker [7]. See page 12 for a definition of this function.

[11] These questions remain open.

[12] We do not need the ‘cantorian’ condition. If there is a strong inaccessible greater than $\aleph_0$ the least such will be cantorian.

[13] In 2009 we still have no consistency results, relative to NF, for any choice principle.

strongest that can be shown consistent relative to NF. Naturally we cannot prove Con(NF) in those theories. The results are as follows:

- Let $Z_{\Delta_0}F_{\Sigma_1}$ be the system obtained from ZF by restricting separation to Δ_0 formulæ and replacement to Σ_1 formulæ. Then in chapter 6 we show

  $$\mathrm{Con(NF)} \to \mathrm{Con}(Z_{\Delta_0}F_{\Sigma_1})$$

- Let Z_n be Z\ P as above with the following additional axioms:

  P_1 If x is the same size as a von Neumann ordinal then $\mathcal{P}(x)$ exists.

  P_2 If x is the same size as a von Neumann ordinal then $\mathcal{P}(x)$ and $\mathcal{P}^2(x)$ exist.

  P_3 If x is the same size as a von Neumann ordinal then $\mathcal{P}(x)$, $\mathcal{P}^2(x)$ and $\mathcal{P}^3(x)$ exist.

  $\vdots$

  P_n If x is the same size as a von Neumann ordinal then $\mathcal{P}(x)$, $\mathcal{P}^2(x)$ $\ldots$ and $\mathcal{P}^n(x)$ exist.

  We show in chapter 7 that, for each $n = 1\ldots$ and each $k = 0\ldots$, NF proves the consistency of Z_n + TC + the existence of the von Neumann ordinals $\omega_0 \ldots \omega_k$.

  By compactness we infer that NF proves the consistency of the union of all the Z_n + TC + the existence of the von Neumann ordinals with concrete subscripts.

  Remark: the construction of these models is possible because we can define membership in the relevant models in such a way that the interpretation of a formula ϕ is stratified *whether ϕ is itself stratified or not.*

I would like to thank Maurice Boffa for all the help he has extended me in the preparation of this work. I would also like to thank the other members of the seminaire NF for fruitful interactions.

Notations and Definitions

V is the universe, $\{x : x = x\}$. $\emptyset$ is the empty set, $\{x : x \neq x\}$. $\bigcup a$ is the sumset of a. (In NF $\bigcap \emptyset = V$ is a *set*). $\mathcal{P}(x)$ is the power set of x. $\iota``x = \{\{t\} : t \in x\}$.

Tuples

The Kuratowski pair $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$ has the disadvantage that, in the expression '$z = \langle x, y\rangle$', 'x', 'y' and 'z' are not all the same type. Since we know (from [7]) that V is infinite, we can partition V into two pieces V_1 and V_2 with bijections $f : V \longleftrightarrow V_1$ and $g : V \longleftrightarrow V_2$. Thus we can take the pair $\langle x, y\rangle$ to be $f``x \cup g``y$. ($f``x$ is of course $\{f(z) : z \in x\}$.) Defining pairs in this way has the advantage that, in the expression '$z = \langle x, y \rangle$', 'x', 'y' and 'z' are all the same type. Quine pairs are obtained in this way: g is defined so that $g(t)$ is obtained from t by adding 1 to every natural number in t and then inserting 0; f is defined so that $f(t)$ is obtained from t by adding 1 to every natural number in t. Unless explicitly stated otherwise our pairs will be Quine pairs.

Cardinals

$A \sim B$ ("A and B are equipollent") says that there is a bijection between A and B[14]. Cardinals are equipollence classes. NC is the set of cardinals. The operation T is defined on cardinals by $T|A| = |\iota``A|$. A cantorian cardinal is the cardinal of a cantorian set. A strongly cantorian cardinal is the cardinal of a strongly cantorian set. $|A| \leq |B|$ says that there is an injection from A to B. NC is not totally ordered by $\leq$; for example $|NO| \not\leq T^2|V|$. The naturals are the cardinals of finite sets. If x is a cardinal of the form $|\iota``A|$, 2^x is $|\mathcal{P}(A)|$, if x is not of this form 2^x is defined to be the empty set. This definition ensures that '$y = 2^x$' is stratified with 'x' and 'y' having the same type.[15]

Ordinals

If R is a well-ordering then $No(R)$ is the ordinal of R, to wit the set of well-orderings isomorphic to R. NO is the set of all ordinals. For any relation R we write '$RUSC(R)$' for $\{\langle \{x\}, \{y\} \rangle : \langle x, y \rangle \in R\}$. We define T on NO by $T(No(R)) := No(RUSC(R))$. $No(R) \leq No(S)$ says that R is isomorphic to an initial segment of S. NO is well-ordered by $\leq$. We say that α is an *initial ordinal* if $(\forall \delta < \alpha)(|\{\beta : \beta < \delta\}| < |\{\beta : \beta < \alpha\}|)$. ω_0 is the first infinite initial ordinal; thereafter ω_{k+1} is the least initial ordinal $> \omega_k$ if there are any. Finally $\aleph_n$ is $\mathrm{Card}(\omega_n)$.[16]

Structures

$\langle A, R \rangle$ is a **structure** if $R \subseteq A \times A$.[17]

$\langle A, R \rangle \simeq \langle B, S \rangle$ says that $\langle A, R \rangle$ and $\langle B, S \rangle$ are isomorphic. $\langle B, \in \rangle$ is the set B equipped with the membership relation. If R is a well-founded extensional relation (a full definition will be given in chapter 2) we define the *domain* $dom(R)$ of R to be $\{a : (\exists b)(\langle a, b \rangle \in R \vee \langle b, a \rangle \in R)\}$. When R is nonempty and $a \in dom(R)$ we say $seg_R(a)$ is the restriction of R to the $\subseteq$-smallest Y such

[14] Hinnion uses a hash sign, but I don't know how to do one in LaTeX!

[15] The significance of this is that it means that the closure of the singleton $\{x\}$ of the cardinal x under $y \mapsto 2^y$ is a set, since it is defined by a closed set abstract: $\bigcap\{A : x \in A \wedge (y \in a \to 2^y \in A)\}$. This object is written '$\Phi(x)$' in Specker [7], where it is an essential gadget in the refutation of AC in NF. See also the definition of cardinal exponentiation in [12].

[16] Hinnion doesn't seem to define this Card() function here, but it's clear that it is the same use of this string as in [4]. If $\alpha = No(R)$ then $card(\alpha)$ is $|dom(R)|$, the cardinality of the domain (carrier set) of R.

[17] Of course he means *binary* structure. However I think that nowhere in this document does he consider any structures that aren't binary, so we can take 'binary' as read.

that $a \in Y$ and $(\forall b \forall c)(b\ Rc \wedge c \in Y \to b \in Y)$. We adopt the convention that when $seg_R(a) = \emptyset$ then $dom(seg_R(a))$ is $\{a\}$. This has the effect that $dom(\emptyset)$ is a singleton—which will be clear from the context. This (slightly ambiguous) definition of domain allows us to work with relations rather than structures and makes for simpler notation.[18]

[18] For more on this see the footnote on p. 22.

Chapter 1

$\mathcal{E}$-structures

1.1 A theorem of Boffa's

Theorem 1 *If NF is consistent, every finite extensional structure has an end-extension that is a model of NF.*

Lemma 2 *If $\langle A, R\rangle$ is a finite extensional structure there is a permutation p of finite support of V_ω such that $\langle A, R\rangle$ is isomorphic to a transitive substructure of $\langle V_\omega, \in_p\rangle$.*

Proof of lemma.

The "$\in_p$" in the statement of the lemma is defined[1] by $x \in_p y$ iff $p(x) \in y$. Since A is finite we may safely suppose that every element of A is of the form $\langle \emptyset, x\rangle$ for some $x \in V_\omega$ and where the pairing function is the Kuratowski pair. Thus $A \subseteq V_\omega$. Let a_R be $\{b \in A : b\, R\, a\}$. If we now set $B := \{a_R : a \in A\}$ we can see that $B \subseteq V_\omega$ and that A and B are disjoint. Now let p be the permutation that swaps x with x_R for $x \in A$ and fixes everything else. An elementary calculation now establishes that $\langle A, R\rangle \simeq \langle B, \in_p\rangle$ and that $\langle B, \in_p\rangle$ is a transitive substructure of $\langle V_\omega, \in\rangle$. ■

Lemma 3 *If $\langle M, S\rangle$ is a model of NF and p a permutation of M of finite support then $\langle M, S_p\rangle$ is also a model of NF.*

Proof of lemma.

S_p is defined here by $x\ S\,_p y \longleftrightarrow p(x)\ S\ y$. We know that if p is a permutation of M that is a set of the model $\langle M, S\rangle$ then $\langle M, S_p\rangle$ is also a model of NF. Clearly if p is a permutation of finite support it has finite graph and is a set of $\langle M, S\rangle$. ■

Proof of theorem:

[1]Beware: this notation is often used for $x \in p(y)$ instead!

If NF is consistent it has a model $\langle M, S\rangle$. Further, $\langle V_\omega, \in\rangle$ is a transitive substructure of $\langle M, S\rangle$ (up to isomorphism). Now let $\langle A, R\rangle$ be a finite structure. By the first lemma there is a permutation p of $\langle V_\omega, \in\rangle$ of finite support such that $\langle A, R\rangle$ is isomorphic to a transitive substructure of $\langle V_\omega, \in\rangle$. From this we infer that $\langle M, S_p\rangle$ is an end-extension of $\langle A, R\rangle$. By the second lemma $\langle M, S_p\rangle$ is a model of NF. ■

Corollary 4 *Every finite structure can be embedded in a model of NF.*

Proof: This follows from theorem 1 and from the fact that every finite structure can be embedded in a finite extensional structure. If $\langle A, R\rangle$ is a finite structure with $A = \{1 \ldots n\}$ (without loss of generality) we consider the extension $\langle B, S\rangle$ of $\langle A, R\rangle$ where $B = A \cup \{n+1, \ldots 2n\}$ and S is $R \cup \{\langle k, k+1\rangle : n+1 \leq k < 2n\} \cup \{\langle k+n, k\rangle : 1 \leq k \leq n\} \cup \{\langle k+n+1, k\rangle : 1 \leq k < n\}$. It is easy to see that $\langle B, S\rangle$ is an extensional structure.

Corollary 5 *If σ is a universal sentence (that is to say a sentence of the form $(\forall x_1 \ldots x_n)\phi$ where ϕ is quantifier-free and has no free variables beyond 'x_1' ... 'x_n') [in the language of set theory], then*

NF $\vdash \sigma$ iff σ is a valid formula of first-order logic.

Proof: Right $\rightarrow$ left is obvious. For the other direction suppose σ is a universal sentence [in the language of set theory] that is not valid. Then there is a countermodel $\langle B, S\rangle$. Let $\neg\sigma$ be $(\exists x_1 \ldots x_n)\neg\phi(x_1 \ldots x_n)$. So we can find witnesses $a_1 \ldots a_n$ for the 'x_i' in B. Now set A to be $\{a_1 \ldots a_n\}$ and procede as above. Set $R = S \restriction A$ so $\langle A, R\rangle \models \neg\phi(a_1 \ldots a_n)$, which is to say $\langle A, R\rangle \models \neg\sigma$. By corollary 4 $\langle A, R\rangle$ has an end-extension $\langle A', R'\rangle$ that is a model of NF. Clearly $\langle A', R'\rangle \models \neg\sigma$. ■

Corollary 6 *Every structure (finite or infinite) embeds in a model of NF.*

Proof:

Let $\langle A, R\rangle$ be a structure. We will define a theory T in the language of NF enriched with constants `a`, `b` ... to denote elements of A. The axioms of T will be `a`$\in$`b` whenever $\langle a, b\rangle \in R$, and `a`$\notin$`b` whenever $\langle a, b\rangle \notin R$. T is consistent; NF + T will be consistent as long as each finite fragment of it is consistent. Let NF + T' be such a finite fragment. T' being finite, it has a model $\langle B, S\rangle$ which is a substructure of $\langle A, R\rangle$. By corollary 4, $\langle B, S\rangle$ can be extended to a model $\langle M, S'\rangle$ of NF $+T'$.

So NF + T is consistent and has a model, and clearly $\langle A, R\rangle$ can be embedded in any such model. ■

1.2 $\mathcal{E}$-structures

Corollary 4 shows that every structure can be embedded in *some* model of NF. One might wonder if every finite structure can be embedded in *every* model of NF. We obtain a positive answer as a corollary of a more general result (theorem 9 and corollary 10.[2]

1.2.1 Preliminary Definitions

In NF we will say that $\langle A, R\rangle$ is a **$\mathcal{E}$-structure** iff there is an injection $f : A \to V$ such that $(\forall a, b \in A)(a\, R\, b \longleftrightarrow f(a) \in f(b))$. We say that A is a **separated** set if $(\forall x, y \in A)(x \notin y)$.

Two Lemmas

Lemma 7 *Every set is the same size as a separated set.*

Proof: Define $g : A \to V$ by $g(a) := \langle a, \emptyset\rangle$. (Here we are using Quine pairs). Clearly g is injective. A is clearly the same size as $B := g``A$. B is a separated set because of the definition of Quine pairs: $(\forall a)(0 \notin \langle a, \emptyset\rangle$ and $(\forall t)(\forall y)(t \in \langle y, \emptyset\rangle \to 0 \in t)$. ■

Lemma 8 *If B is a strongly cantorian separated set and $R \subseteq B \times B$ then $\langle B, R\rangle$ is a $\mathcal{E}$-structure.*

Proof:

Let a_x be $\{t : x \in t\}$. The expression[3] '$y = a_x$' is stratified with 'y' two types higher than 'x'. Define $f : B \to V$ by $f(x) := \{x\} \cup (\bigcup\{a_y : y\, R\, x\})$. It's easy to see that f is not defined by a stratified formula. No difficulty arises because B is strongly cantorian: $\iota\restriction B$ is the obvious bijection between B and $\iota``B$. Let us abbreviate this to 'ϕ', and let $RUSC(\phi)$ be ψ. Now we can supply a stratified definition for f: for $x \in B$, $f(\phi(\psi(\{x\}))) = \{\phi(\{x\})\} \cup (\bigcup\{a_y : y\, R\, x\})$.

Now we show that, for all s and t in B, $s \in f(t) \longleftrightarrow s = t$.

Right $\to$ Left follows easily from the definition of f. For the other direction if $s \in f(t)$ we must have $s = t$ or $s \in \bigcup\{a_z : z\, R\, t\}$. This second possibility is excluded because B is separated. ■

Next we establish that f is injective. Suppose $f(s) = f(t)$. By extensionality we have $(\forall x)(x \in f(s) \longleftrightarrow x \in f(t))$. Since $s \in f(s)$ we have $s \in f(t)$ and thence $s = t$. ■

Thirdly we want $(\forall s, t \in B)(sRt \longleftrightarrow f(s) \in f(t))$. If $s\, R\, t$ then (by definition of f) $a_s \subset f(t)$. Since $s \in f(s)$ then (by definition of a_s) we have $f(s) \in a_s$,

[2] Later work shows that every countable structure embeds in every model of NFO. See [14].

[3] This device is normally written '$B(x)$', the 'B' alluding to Boffa who noticed that it is an $\in$-isomorphism. It appears in Hailperin's finite axiomatisation in [5]. Quine noticed it too, and Whitehead suggested to him that $\{y : y \in x\}$ should be called the *essence* of x. I don't think the notation was in use at the time Hinnion was preparing his thesis.

whence $f(s) \in f(t)$. For the other direction, suppose $f(s) \in f(t)$. Then $f(s) \neq t$ because $s \in f(s)$ and $s \notin t$ because B is separated. Accordingly there is $z \in B$ such that zRt and $f(s) \in a_z$. So $(\exists z \in B)(zRt \wedge z \in f(s))$. Now $z \in f(s)$, whence $z \in B$ and $z = s$. We conclude that sRt. ■

Theorem 9 *NF $\vdash$ Every structure $\langle A, R\rangle$, where A is strongly cantorian, is a $\mathcal{E}$-structure.*

Proof:
By lemma 7 there is a structure $\langle B, S\rangle \simeq \langle A, R\rangle$ with B separated. By lemma 8 $\langle B, S\rangle$ is an $\mathcal{E}$-structure, so $\langle A, R\rangle$ is an $\mathcal{E}$-structure too. ■

Corollary 10 *Every finite structure embeds in every model of NF.*

Proof: If $\langle A, R\rangle$ is a finite structure we may assume without loss of generality that A is $[1, n]$ the first n natural numbers. A is strongly cantorian, since the bijection $\iota \restriction A$ can be given explicitly: $\{\langle 1, \{1\}\rangle, \langle 2, \{2\}\rangle, \langle 3, \{3\}\rangle \ldots\}$.[4] So $\langle A, R\rangle$ is an $\mathcal{E}$-structure. So $\langle A, R\rangle$ can be embedded in every model of NF. ■

Corollary 11 *Let σ be a universal sentence [in the language of set theory], and assume NF consistent. Then $NF \vdash \neg\sigma$ iff σ is not a valid formula of first-order logic.*

Proof: Suppose that NF is consistent, and that $\langle M, S\rangle$ is a model of it. If NF refutes σ we must have $\langle M, S\rangle \models \neg\sigma$. If $\neg\sigma$ we know there must be a structure $\langle A, R\rangle$ which satisfies $\neg\sigma$. If σ is $(\forall x_1 \ldots x_n)\phi$ then there are $a_1 \ldots a_n$ in A such that $\langle A, R\rangle \models \neg\phi(a_1 \ldots a_n)$. Let A' be $\{a_1 \ldots a_n\}$ and let R' be $R \restriction A'$. Evidently $\langle A', R'\rangle \models \neg\sigma$. The structure $\langle A', R'\rangle$—being finite—embeds into every model of NF (corollary 10), so every model of NF is a model of $\neg\sigma$. ■

Corollary 12 *NF decides all universal sentences.*

Proof: By corollaries 5 and 10 we have: if σ is a universal sentence then NF$\vdash \sigma$ iff σ is valid and NF$\vdash \neg\sigma$ iff σ. Since σ either is or is not valid we infer that NF either proves σ or refutes it. ■

1.2.2 Remarks

The Hypotheses of theorem 9

Can we weaken the hypotheses of theorem 9? The following propositions answer this question.

[4] The point is that $\langle A, R\rangle$ is a finite structure in an external sense, so its cardinal number is a *concrete* natural number. Every concrete natural number is strongly cantorian.

PROPOSITION 13 *Suppose $\langle A, <\rangle$ is a well-ordering. If $\langle A, <\rangle$ is an $\mathcal{E}$-structure A must be strongly cantorian.*

Proof: If $\langle A, <\rangle$ is an $\mathcal{E}$-structure there is an injection $f : A \to V$ such that $(\forall a, b \in A)(a < b \longleftrightarrow f(a) \in f(b))$. Let $B = f``A$. Suppose A has a maximum element c. We have $\{f(a)\} = f(a+1) \setminus f(a) \setminus (V \setminus B)$ whence $a < c$ and $a+1$ is the successor of a in $\langle A, <\rangle$. If $x \in f(a+1) \setminus f(a) \setminus (V \setminus B)$ then $x \in B$. So there is $b \in A$ such that $x = f(b)$. Therefore $f(b) \in f(a+1)$ whence $b < a+1$. Since $f(b) \in f(a)$ we have $b \not< a$. Therefore $b = a$. stuff to sort out

PROPOSITION 14 *NF $\vdash$ Not every structure is an $\mathcal{E}$-structure.*

Proof: Consider the structure $\langle NO, <\rangle$ where $<$ is the usual well-ordering of the ordinals. If every structure were an $\mathcal{E}$-structure then NO would be strongly cantorian, and it isn't. ■

PROPOSITION 15 *If NF is consistent it does not prove that every structure whose domain is cantorian is a $\mathcal{E}$-structure.*

Proof: Suppose not, and that every structure whose domain is cantorian is a $\mathcal{E}$-structure. $\langle \mathbb{N}, <\rangle$ is a structure with cantorian domain. Since $<$ is a well-ordering it would follow that $\langle \mathbb{N}, <\rangle$ is a $\mathcal{E}$-structure and strongly cantorian. But it is known that "$\mathbb{N}$ is strongly cantorian" (the Axiom of Counting) is independent of NF[5]. ■

A converse

We have shown that NF $\vdash$ For all structures $\langle A, R\rangle$ if A is strongly cantorian then $\langle A, R\rangle$ is an $\mathcal{E}$-structure. We will now show that NF $\vdash$ it is not the case that, for all structures $\langle A, R\rangle$, if $\langle A, R\rangle$ is an $\mathcal{E}$-structure then A is cantorian. It will suffice to consider the structure $\langle V, \emptyset\rangle$. We know from lemma 7 that there is a separated set B the same size as V. Manifestly $\langle V, \emptyset\rangle \simeq \langle B, \in\rangle$, so $\langle V, \emptyset\rangle$ is an $\mathcal{E}$-structure. But V is not cantorian.

[5] This is in Orey [17].

Chapter 2

The Structure $\langle BF, \mathcal{E}\rangle$

2.1 Introduction

We are going to define in NF a structure $\langle BF, \mathcal{E}\rangle$ which will be a transitive [end-]extension of the structure $\langle NO, <\rangle$ where $<$ is the usual strict order on the ordinals. We will use it to extend a result of Henson ([1] theorem 2.4). Furthermore, various substructures of $\langle BF, \mathcal{E}\rangle$ will afford us models and interpretations of fragments of ZF in suitable extensions of NF.

2.2 Definitions

Definition 16 *A structure $\langle A, R\rangle$ is well-founded and extensional [a BFEXT] iff*

1. $dom(R) = A$;
2. $(\forall B)((B \neq \emptyset \wedge B \subseteq A) \to (\exists b \in B)(\forall c \in B)(\neg(c\, R\, b)))$;
3. $(\forall a, b \in A)(a = b \longleftrightarrow (\forall c)(c\, R\, a \longleftrightarrow c\, R\, b))$.

Definition 17 *A relation R is* **well-founded extensional** *iff $\langle dom(R), R\rangle$ is a BFEXT.*

Definition 18 *If R is well-founded extensional and $B \subseteq dom(R)$ is nonempty then we say b is minimal in B iff $(\forall c \in B)(\neg c\, R\, b)$.*

Definition 19 *If $\langle A, R\rangle$ is a BFEXT then for each $a \in A$ we define $seg_R(a)$ to be $R{\restriction}\bigcap\{B : B \subseteq A \wedge a \in B \wedge (\forall b \in B)(\forall c \in A)(c\, R\, b \to c \in B)\}$.*

Definition 20 $\Omega :=$
$\{R : R$ *is a well-founded extensional relation with* $(\exists a \in dom(R))(R = seg_R(a)\}$.

It's easy to check that, if $R \in \Omega$, then $(\exists! a)(R = seg_R(a))$. This element will be notated '$\mathbb{1}_R$'.[1]

Definition 21 *If* $R \in \Omega$ *then* $\mathcal{T}(R)$*, the* **type** *of* R*, is* $\{S \in \Omega : S \simeq R\}$.[2]

Definition 22 $BF = \{\mathcal{T}(R) : R \in \Omega\}$.

Definition 23 $\mathcal{T}(R)\ \mathcal{E}\ \mathcal{T}(S)$ *iff* $(\exists a \in dom(S))(R \simeq seg_s(a) \wedge a\ S\ (\mathbb{1}_S))$.

We can now define on BF an analogue of T on the ordinals:

Definition 24 $T\mathcal{T}(R) := \mathcal{T}(RUSC(R))$.

2.3 Remarks

The possibly obscure difference between '$seg_R(a)$' where $R \in \Omega$ and '$seg_{\leq}(\alpha)$' which last denotes $\{\langle \beta, \delta\rangle : \beta, \delta \in NO \wedge \beta \leq \delta < \alpha\}$ (where α is an ordinal and $\leq$ is the usual order on NO) can always be clarified in context.

Each ordinal corresponds to a unique element of BF. If α is a finite ordinal set $f(\alpha) := \mathcal{T}(R')$, where R' is the strict well-order corresponding to some—any—R in α. If α is not finite we reason as follows. α is $No(R)$ for some well-order R and we can define a relation S_ζ to be $R' \cup \{\langle x, \zeta\rangle : x \in dom(R)\}$ where ζ is some object not in $dom(R)$. (There will always be such a ζ because V cannot be well-ordered.) We now take $f(\alpha)$ to be $\mathcal{T}(S_\zeta)$ and it is clear that $f(\alpha)$ does not depend on the choice of ζ. f is now an injection $NO \to BF$ because:

$$(\forall \alpha, \beta \in NO)(\alpha < \beta \longleftrightarrow f(\alpha)\ \mathcal{E}\ f(\beta)) \tag{2.1}$$

$$(\forall \alpha \in NO)(f(T\alpha) = Tf(\alpha)) \tag{2.2}$$

These two observations are easy to check, as is the following fact. $\langle NO, <\rangle$ is, up to isomorphism, a transitive substructure of $\langle BF, \mathcal{E}\rangle$, and the T operation on ordinals is the restriction to the ordinals of the T operation on BF.[3]

[1]Important notation this. Ω is the set of well-founded relations that look like set pictures of well-founded sets. In another notation, Ω is the set of BFEXTs.

[2]Thus the *types* which will be so useful to us are isomorphism classes of *relations* not of *structures*.

[3]This is a bit messy, but a certain amount of mess is unavoidable. BFEXTS (or at least elements of Ω) have top elements (in some sense!) so no well-ordering of *limit* length can be a BFEXT—since it lacks a top element. So NO is never going to be an initial segment of BF. Should it not at least be a subset? The injection that Hinnion provides from NO to BF is not the inclusion embedding because Hinnion has taken his well-orderings to be special kinds of *reflexive* total orderings. Why does he do this? If we construe well-orderings as special kinds of *strict* total orders—which is what we should do if we want well-orderings to be special kinds of BFEXTS—-then we cannot distinguish well-orders of singletons from each other or from the (empty) well-order of the empty set, since all are encoded by the one empty set of ordered pairs. It is true that we can think of well-orders and BFEXTS as structures

2.4 Propositions

It is easy to see that if R and S are isomorphic BFEXTs then the isomorphism is unique.[4]

PROPOSITION 25
If R is a BFEXT then $(\forall a, b \in dom(R))(seg_R(a) \simeq seg_R(b) \longleftrightarrow a = b)$.

Proof:
Let g be the hypothesised isomorphism between $seg_R(a)$ and $seg_R(b)$. We will prove [by R-induction] that g is the identity. Suppose not, and let c be an R-minimal element moved by g. By minimality we have $(\forall b)(b\ R\, c \to g(b) = b)$. Since g is an isomorphism we have $(\forall b)(b\ R\, c \longleftrightarrow g(b)\ R\, g(c))$. Therefore $(\forall b)(b\ R\, c \longleftrightarrow b\ R\, g(c))$. Now, by extensionality of R, we infer $c = g(c)$ contradicting assumption.

PROPOSITION 26 $\langle BF, \mathcal{E}\rangle$ *is a well-founded extensional structure.*

Proof:
There are three clauses to check in definition 16.

1. is satisfied because $dom(\mathcal{E}) = BF$.

2. Let A be a nonempty subset of BF, so we can find $\mathfrak{a} \in A$. If $\mathfrak{a} = \mathcal{T}(R)$ let A' be $\{b \in dom(R) : \mathcal{T}(seg_R(b)) \in A\}$. We have $\mathbb{1}_R \in A'$ so A' is nonempty, and is a subset of $dom(R)$. So there must be an R-minimal b in A'. $\mathcal{T}(seg_R(b))$ is now $\mathcal{E}$-minimal in A.

3. Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are in BF and $(\forall \mathfrak{c} \in BF)(\mathfrak{c}\ \mathcal{E}\ \mathfrak{a} \longleftrightarrow \mathfrak{c}\ \mathcal{E}\ \mathfrak{b})$. We want to show that $\mathfrak{a} = \mathfrak{b}$. $\mathfrak{a}$ must be $\mathcal{T}(R)$ and $\mathfrak{b}$ must be $\mathcal{T}(S)$ for some R and S. Since $\mathfrak{c}\ \mathcal{E}\ \mathfrak{a}$ there is $x \in dom(R)$ such that $x\ R\,(\mathbb{1}_R)$ and $\mathfrak{c} = seg_R(x)$. From this it follows that $seg_R(x) \simeq seg_S(y)$. The map $F : \{x : x\ R\,(\mathbb{1}_R)\} \to \{y : y\ S\,(\mathbb{1}_S)\}$ given by $F(x) = y$ iff $seg_R(x) = seg_S(y)$ is a bijection by proposition 26. Manifestly $R \simeq S$ by means of the isomophism

$$\{\langle \mathbb{1}_R, \mathbb{1}_S\rangle\} \cup \bigcup\{\psi : (\exists x)(x\ R\,(\mathbb{1}_R) \wedge seg_R(x) \simeq_\psi seg_S F(x))\}$$

■

PROPOSITION 27 $(\forall \mathfrak{a}, \mathfrak{b} \in BF)(\mathfrak{a}\ \mathcal{E}\ \mathfrak{b} \longleftrightarrow T\mathfrak{a}\ \mathcal{E}\ T\mathfrak{b})$.

Proof:
If $\mathfrak{a} = \mathcal{T}(R)$ and $\mathfrak{b} = \mathcal{T}(S)$, then[5]

---

not relations, and that takes care of the problem with the singleton well-orders and the empty set. It would also ensure that the injection from NO to BF is the inclusion embedding, but nothing can change the fact that NO is not going to be an initial segment of BF, because of the existence of limit ordinals.

[4] We prove this by well-founded induction on R.

[5] He's re-used the letter 'b'. Better sort this out ...

$$\begin{aligned}
\mathfrak{a}\,\mathcal{E}\,b &\longleftrightarrow ((\mathfrak{b}\ \ S\ \ (\mathbb{1}_S)) \wedge (R \simeq seg_S(\mathfrak{b})))\\
&\longleftrightarrow (\exists b)((\{b\}\ RUSC(S)\ (\mathbb{1}_{RUSC(S)})) \wedge (RUSC(R)\ \simeq\ (seg_{RUSC(S)}\{b\})\\
&\longleftrightarrow \mathcal{T}(RUSC(R))\ \mathcal{E}\ \mathcal{T}(RUSC(S))\\
&\longleftrightarrow T\mathfrak{a}\ \mathcal{E}\ T\mathfrak{b}
\end{aligned}$$

We assert the following without proof:

Proposition 28

1. *$(\forall \mathfrak{a}, \mathfrak{b} \in BF)(\mathfrak{a}\ \mathcal{E}\ \mathfrak{b} \longleftrightarrow \mathcal{T}(seg_{\mathcal{E}}(\mathfrak{a}))\ \mathcal{E}\ \mathcal{T}(seg_{\mathcal{E}}(\mathfrak{b})))$;*
2. *$(\forall \mathfrak{a}, \mathfrak{b} \in BF)(\mathfrak{a} \in dom(seg_{\mathcal{E}}(T\mathfrak{b})) \to (\exists \mathfrak{c} \in dom(seg_{\mathcal{E}}(\mathfrak{b})))(\mathfrak{a} = T\mathfrak{c}))$;*
3. *$(\forall \mathfrak{a}, \mathfrak{b} \in BF)(\mathfrak{a}\ \mathcal{E}\ T\mathfrak{b} \to (\exists \mathfrak{c})(\mathfrak{c}\ \mathcal{E}\ \mathfrak{b} \wedge \mathfrak{a} = T\mathfrak{c}))$.*

Proposition 29 *$(\forall \mathfrak{a} \in BF)(\mathcal{T}(seg_{\mathcal{E}}(\mathfrak{a})) = T^2\mathfrak{a})$.*

Proof: If $A = \{\mathfrak{a} \in BF : \mathcal{T}(seg_{\mathcal{E}}(\mathfrak{a})) \neq T^2\mathfrak{a}\}$ is nonempty then it has a minimal element $\mathfrak{a}_0$.

- Suppose $\mathfrak{c}\ \mathcal{E}\ T^2\mathfrak{a}_0$. By proposition 28 (3) there is $\mathfrak{b}$ in BF such that $\mathfrak{c} = T^2\mathfrak{b}$. We can use proposition 27 to infer from $T^2\mathfrak{b}\ \mathcal{E}\ T^2\mathfrak{a}_0$ that $\mathfrak{b}\ \mathcal{E}\ \mathfrak{a}_0$. Since $\mathfrak{a}_0$ is minimal in A we have $T^2\mathfrak{b} = \mathcal{T}(seg_{\mathcal{E}}(\mathfrak{b}))$. Now, by proposition 28 (1), $\mathfrak{b}\ \mathcal{E}\ \mathfrak{a}_0 \to \mathcal{T}(seg_{\mathcal{E}}(\mathfrak{b}))\ \mathcal{E}\ \mathcal{T}(seg_{\mathcal{E}}(\mathfrak{a}_0))$. Since $\mathfrak{c} = T^2\mathfrak{b} = \mathcal{T}(seg_{\mathcal{E}}(\mathfrak{b}))$ we have $\mathfrak{c}\ \mathcal{E}\ seg_{\mathcal{E}}(\mathfrak{a}_0)$.
- Suppose $\mathfrak{c}\ \mathcal{E}\ \mathcal{T}(seg_{\mathcal{E}}(\mathfrak{a}_0))$. Then there is a $\mathfrak{b}\ \mathcal{E}\ \mathfrak{a}$ such that $\mathfrak{c} = \mathcal{T}(seg_{\mathcal{E}}(\mathfrak{b}))$. Since $\mathfrak{a}_0$ is minimal in A we have $T^2\mathfrak{b} = \mathcal{T}(seg_{\mathcal{E}}(\mathfrak{b}))$. By proposition 27 $\mathfrak{b}\ \mathcal{E}\ \mathfrak{a}_0$ implies $T^2\mathfrak{b}\ \ \mathcal{E}\ T^2(\mathfrak{a}_0)$.

The two bullet points establish that $(\forall \mathfrak{c} \in BF)[\mathfrak{c}\ \mathcal{E}\ T^2\mathfrak{a}_0 \longleftrightarrow \mathfrak{c}\ \mathcal{E}\ \mathcal{T}(seg_{\mathcal{E}}(\mathfrak{a}_0))]$.
■

2.5 Generalisation of a theorem of Henson

2.5.1 Preliminary Definitions

Definition 30 *'$\langle A, R\rangle \simeq \langle B, \in\rangle$' is short for the conjunction of*
(1) $\in\!\restriction\! B$ (the restriction of $\in$ to B) is a set;
(2) There is a bijection $f : A \longleftrightarrow B$ such that $x\, R\, y \longleftrightarrow f(x) \in f(y)$.

Then we say "$\langle A, R\rangle$ is an $\mathcal{E}$-transitive structure" iff there is a transitive set B such that $(\forall xy)(\langle A, R\rangle \simeq \langle B, \in\rangle)$.

2.5.2 A Theorem of Henson's

Theorem 31 *Let T be an invariant extension of NF. If T is consistent then so too is $T+$ "Every structure $\langle dom(R), R\rangle$ where R is a strict[6] well-ordering*

[6]Remember that for Hinnion well-orderings are reflexive.

and $dom(R)$ is strongly cantorian is an $\mathcal{E}$-transitive structure" is also consistent.

In ZF we have the principle of contraction[7]: Every well-founded extensional structure $\langle A, R\rangle$ is isomorphic to the membership relation on a transitive set B. We say B is the **contraction** of A. In particular every well-ordering is isomorphic to a von Neumann ordinal. The result which follows generalises theorem 31. It says that the contraction principle holds for strongly cantorian BFEXTs.

Theorem 32 *Let T be a consistent invariant extension of NF. Then NF + "every strongly cantorian well-founded extensional structure is a transitive $\mathcal{E}$-structure" is also consistent.*

Proof:

Our proof will follow closely the proof of the corresponding result of Henson ([1] theorem 2.4). It suffices to replace 'NO' and '$<$' in Henson's proof by 'BF' and '$\mathcal{E}$' respectively. We need two lemmas:

Lemma 33 *If T is a consistent invariant extension of NF then we can consistently add to it the following formula:*

$$(\forall A, R)(A = dom(R) \wedge\ R\ \in \Omega \wedge stcan(A) \to (\exists B \subseteq \mathcal{P}(B))(\langle A, R\rangle \simeq \langle B, \ni\!\!\!\!\in B\rangle))$$

... which we will abbreviate to 'ϕ'.

Proof:

If we use the [Rieger-Bernays] permutation method (see [1] theorem 1.5 for example) we need only find a permutation p such that ϕ_p. From that it will follow that $\mathrm{Con}(T) \to \mathrm{Con}(T + \phi)$. We define $x \in_p y \longleftrightarrow x \in p(y)$. With the help of theorem 1.2 of [1] we can verify that ϕ_p is equivalent to

$$(\forall A)(\forall R)(stcan(A) \wedge A = dom(R) \to$$

$$(\exists B)(\exists f : A \longleftrightarrow B) \bigwedge \left(\begin{array}{c} (\forall ab)((a \in p(b) \wedge b \in B) \to a \in B) \\ (\forall x, y \in A)(x\ R\ y \longleftrightarrow f(x) \in p(f(y))) \end{array} \right)$$

Let p be the permutation defined[8] by

$$\prod_{\mathfrak{a} \in BF} (T\mathfrak{a}, \{\mathfrak{b} \in BF : \mathfrak{b}\ \mathcal{E}\ \mathfrak{a}\})$$

We must show that $T \vdash \Phi_p$.

Suppose $A = dom(R)$ with A strongly cantorian and $R \in \Omega$.

If $\mathfrak{a}_0 = \mathcal{T}(R)$ then—since A is strongly cantorian—we have $RUSC(R) \simeq R$ and therefore $T\mathfrak{a}_0 = \mathfrak{a}_0$. It follows (by proposition 29) that $\mathfrak{a}_0 = T^2\mathfrak{a}_0 =$

[7] He means: the Mostowski collapse lemma.

[8] I have written this permutation as a product of transpositions, where we notate transpositions with round parentheses and commas—*translator's note.*

$\mathcal{T}(seg_{\mathcal{E}}(\mathfrak{a}_0))$. So $R \simeq seg_{\mathcal{E}}(\mathfrak{a}_0)$. Let f be the relevant isomorphism (we noted the uniqueness of f on p 23); f is a bijection between A and $B = seg_{\mathcal{E}}(\mathfrak{a}_0)$.

Next we show that $(\forall \mathfrak{a}, \mathfrak{b})((\mathfrak{a} \in p(\mathfrak{b}) \wedge \mathfrak{b} \in B) \to \mathfrak{a} \in B)$. Since A is strongly cantorian we have $(\forall x \in dom(R))(RUSC(seg_R(x)) \simeq seg_R(x))$ which entails that $(\forall y \in dom(seg_{\mathcal{E}}(\mathfrak{a}_0))(Ty = y)$. If $\mathfrak{b} \in B$ we have $T\mathfrak{b} = \mathfrak{b}$. '$\mathfrak{a} \in p(\mathfrak{b})$' becomes $\mathfrak{a} \in p(T\mathfrak{b}) = \{\mathfrak{c} : \mathfrak{c}\ \mathcal{E}\ \mathfrak{b}\}$, which is to say $\mathfrak{a}\ \mathcal{E}\ \mathfrak{b}$. Since $\mathfrak{b} \in seg_{\mathcal{E}}(\mathfrak{a}_0)$ we infer $\mathfrak{a} \in B$.

Finally we want $(\forall x, y \in A)(x\ R\ y \longleftrightarrow f(x) \in_p f(y))$. Since f is the unique isomorphism $R \simeq seg_{\mathcal{E}}(\mathfrak{a}_0)$ we have $x\ R\ y \longleftrightarrow f(x)\ \mathcal{E}\ f(y))$. Since $f(y) \in B$ and $(\forall z \in B)(Tz = z)$ (*vide* preceding paragraph) we have $Tf(y) = f(y)$. Therefore

$$
\begin{aligned}
x\ R\ y &\longleftrightarrow f(x)\ \mathcal{E}\ f(y) \\
&\longleftrightarrow f(x) \in \{\mathfrak{c} : \mathfrak{c}\ \mathcal{E}\ f(y)\} \\
&\longleftrightarrow f(x) \in p(Tf(y)) \\
&\longleftrightarrow f(x) \in_p f(y)
\end{aligned}
$$

■

Lemma 34 *Every well-founded extensional structure $\langle A, R\rangle$ with A strongly cantorian has an end-extension $\langle B, S\rangle$ with $S \in \Omega$ and B strongly cantorian.*

Proof:

If A is strongly cantorian it is not the universe, so we can find $\zeta \in V \setminus A$. If we set $S :=\ R \cup \{\langle a, \zeta\rangle : a \in A\}$ and $B := A \cup \{\zeta\}$ we observe that $S \in \Omega$ and that $\langle B, S\rangle$ is an end-extension of $\langle A, R\rangle$. ■

Proof of theorem 32

We know from lemma 33 that if T is a consistent invariant extension of NF then $T + \phi$ is consistent. Working in $T + \phi$, we consider a well-founded extensional structure $\langle A, R\rangle$ with A strongly cantorian. By lemma 34 $\langle A, R\rangle$ has an end-extension $\langle B, S\rangle$ with B strongly cantorian. Accordingly, in $T + \phi$, we prove that there is a transitive set C with $\langle B, S\rangle \simeq \langle C, \in\rangle$. Let g be the isomorphism. Evidently g"A is a transitive set and witnesses the fact that $\langle A, R\rangle$ is an $\mathcal{E}$-transitive structure. ■

2.6 Consequences

Theorem 32 is a [partial] response to the question "Can we consistently add to NF or any of its invariant extensions axioms affirming the existence of the sets ω, ζ or V_ω? (Here ω is the von Neumann ω, ζ is the set of Zermelo naturals where $0 := \emptyset$ and $n + 1 := \{n\}$.) These three sets are defined by unstratified formulæ, so if NF proves their existence it does not do so in any straightforward way. Let "$\exists\omega$', '$\exists\zeta$' and '$\exists V_\omega$' be the three axioms asserting the existence of these sets.

PROPOSITION 35 *If NF + Axiom of Counting is consistent so is NF + $\exists\omega$ + $\exists\zeta$ +$\exists V_\omega$.*

Proof:

T = NF + the Axiom of Counting is an invariant extension of NF[9] so we can apply theorem 32: if NF + the Axiom of Counting is consistent we can consistently add to it the assertion that every well-founded extensional structure whose domain is strongly cantorian is $\mathcal{E}$-transitive. This extra assertion will imply all three of $\exists\omega$, $\exists\zeta$ and $\exists V_\omega$.

To deduce $\exists\zeta$ consider the structure $\langle \mathbb{N}, \{\langle n, n+1\rangle : n \in \mathbb{N}\}\rangle$. It follows from the Axiom of Counting that $\mathbb{N}$ is strongly cantorian so this structure has a strongly cantorian domain and is $\mathcal{E}$-transitive. Let g be the collapsing map. Clearly $g(n+1) = \{g(n)\}$ so $g``\mathbb{N}$ is the Zermelo naturals as desired.

The proofs of $\exists\omega$ and $\exists V_\omega$ are similar. For the first, take the transitive collapse of $\langle \mathbb{N}, <_{\mathbb{N}}\rangle$ and for $\exists V_\omega$ consider the set $\{\mathcal{T}(R) \in BF : |dom(R)| < \aleph_0\}$ equipped with the restriction of $\mathcal{E}$. ■

It's worth noting that NF $\cup$ $\{\neg\exists\omega, \neg\exists\zeta, \neg\exists V_\omega\}$ is consistent if NF is. This is because NF $+\exists\omega \vdash$ Axiom of Counting, NF + $\exists\zeta \vdash$ Axiom of Counting and $\exists V_\omega \vdash$ Axiom of Counting.[10] For example, let us derive the Axiom of Counting from $\exists\omega$. Consider the function $f : \mathbb{N} \to V$ defined by $f(0) := \emptyset$ and $f(n+1) := f(n) \cup \{f(n)\}$. This gives us a map $F : \omega \longleftrightarrow \iota``\omega$ defined by $F(n) := f(n+1) \setminus f(n)$. Evidently $\{f(n)\} = f(n+1) \setminus f(n)$ so $\omega$ is strongly cantorian. Evidently $\omega = f``\mathbb{N}$ so $|\omega| = |\mathbb{N}|$ and $\mathbb{N}$ is strongly cantorian too.[11]

[9] Hinnion doesn't prove this fact, which was well-established by the time this thesis was written. It's in the work of Henson, and later Pétry. Roughly every assertion in the theory of isomorphism types is invariant. Certainly all of cardinal and ordinal arithmetic is invariant.

[10] The allegation is correct but the explanation given is not. The consistency of NF + $\{\neg\exists\omega, \neg\exists\zeta, \neg\exists V_\omega\}$ follows from the fact that it is consistent with NF that there should be a finite set x with $\mathcal{P}(x) \subset x$. In any such model every well-founded set is finite, so ω, ζ and V_ω—being well-founded infinite sets—must be missing from it. It is true that, as Hinnion writes, the Axiom of Counting follows from $\exists\omega$, but no proof is known that the Axiom of Counting follows from the existence of V_ω or of ζ, and it seems highly unlikely that it should should follow from either or even their conjunction. The assertion "both V_ω and the graph of the set-theoretic rank function restricted to V_ω exist" has the same consistency strength as "$(\forall n \in Nn)(n \leq Tn)$"—which Hinnion later on in this document (see the footnote on page 37) introduces under the name 'AxCount$_\leq$'—and this is believed to be much weaker than the Axiom of Counting. See [15]. Indeed $\exists\zeta$ appears so weak that attempts have even been made to prove its consistency relative to plain NF by permutation methods, but so far without success.

[11] This looks a bit garbled to me: why is the graph of f a set? But the result is correct.

Chapter 3

The notion of *Model* in NF

3.1 Models in NF

3.1.1 Defining formulæ in NF

The symbols which go into formulæ in NF (or ZF) are

$$(,), \in, \exists, \neg, \vee$$

and the variables 'x_0', 'x_1', 'x_2' ...

Symbols can be encoded by natural numbers, for example in the following scheme: '$\in$' $\mapsto 1$, '$=$' $\mapsto 3$, '$\exists$' $\mapsto 5$, '$\neg$' $\mapsto 7$, '$\vee$' $\mapsto 9$, '(' $\mapsto 11$, ')' $\mapsto 13$ and 'x_i' $\mapsto 2i$. The code for a symbol ζ will be written $\ulcorner\zeta\urcorner$. The **length** of a formula is the number of [tokens of] signs that compose it. The **code** of a formula ϕ of length n—written $\ulcorner\phi\urcorner$ [overloading the corners]—is a function f from $[1, n]$ where $f(i)$ is the ith symbol of ϕ.

CF is to be the set of codes of formulæ. It has a stratified definition as a subset of $\mathbb{N}^{<\omega}$, the set of finite sequences of naturals. We can also define a function $\texttt{VL}$: $CF \to \mathcal{P}(\mathbb{N})$ sending each (code of a) formula to the set of (codes of) variables free in that formula. We can give a recursive definition of $\texttt{VL}$:[1]

Definition 36 $\texttt{VL}(\ulcorner x_i \in x_j\urcorner) = \texttt{VL}(\ulcorner x_i = x_j\urcorner) = \{2i, 2j\}$;
$\texttt{VL}(\ulcorner\phi \vee \psi\urcorner) = \texttt{VL}(\ulcorner\phi\urcorner) \cup \texttt{VL}(\ulcorner\psi\urcorner)$;
$\texttt{VL}(\ulcorner(\exists x_i)\psi\urcorner) = \texttt{VL}(\ulcorner\psi\urcorner) \setminus \{2i\}$;
$\texttt{VL}(\ulcorner\neg\psi\urcorner) = \texttt{VL}(\ulcorner\psi\urcorner)$.

The formula '$a = \texttt{VL}(b)$' is stratified with 'a' and 'b' receiving the same type.[2] We will omit corner quotes when this can be done without causing confusion.

[1] '**V**ariable **L**ibre': *free variable* in French.

[2] In virtue of the fact that all its free variables have the same type we say that **homogeneous**.

3.1.2 Precise definition of *model*

Suppose $\mathfrak{M} = \langle A, R \rangle$. For $\phi \in CF$ we can define $S(\phi, \mathfrak{M})$ to be $\mathtt{VL}(\phi) \to A$, the set of functions from $\mathtt{VL}(\phi)$ to A. If $\mathtt{VL}(\phi) = \{2i_1, 2i_2, 2i_3, 2i_n\}$ and $s \in S(\phi, \mathfrak{M})$ then s is of the form

$$\{\langle 2i_1, a_1 \rangle, \langle 2i_2, a_2 \rangle, \ldots \langle 2i_n, a_n \rangle\}$$

To keep things simple let us write s as

$$\{(x_1, a_1)(x_2, a_2) \ldots (x_n, a_n)\}$$

One can define the set $E(\phi, \mathfrak{M})$ of sequences from A that satisfy ϕ. We do it by recursion and in such a way that '$x = E(\phi, \mathfrak{M})$' is stratified with '$x$' one type higher than '$\phi$' and '$\mathfrak{M}$'.

Definition 37

$E(x_i \in x_j, \mathfrak{M}) := \{[(x_i, a), (x_j, b)] : a\, R\, b\}$
$E(x_i = x_j, \mathfrak{M}) := \{[(x_i, a), (x_j, b)] : a = b\}$
$E(\neg\phi, \mathfrak{M}) := \{s \in S(\phi, \mathfrak{M}) : s \notin E(\phi, \mathfrak{M})\}$
$E(\phi \vee \psi, \mathfrak{M}) := \{r \cup s \in S(\phi \vee \psi, \mathfrak{M}) : (r \in E(\phi, \mathfrak{M}) \vee s \in E(\psi, \mathfrak{M})) \wedge (\forall k \in \mathtt{VL}(\phi) \cap \mathtt{VL}(\psi))(r(k) = s(k))\}$
If $2i \in \mathtt{VL}(\phi)$ *then*
$E(\exists x_i \phi, \mathfrak{M}) := \{s \in S(\exists x_i \phi, \mathfrak{M}) : (\exists a \in A)(s \cup \{\langle 2i, a \rangle\} \in E(\phi, \mathfrak{M})\}$
If $2i \notin \mathtt{VL}(\phi)$ *then* $E(\exists x_i \phi, \mathfrak{M}) := E(\phi, \mathfrak{M})$

More Definitions

Definition 38

- *When* $\mathfrak{M} = \langle A, R \rangle$, $\phi \in CF$ *and* $s \in S(\phi, \mathfrak{M})$ *we write '*$\mathfrak{M} \models \phi \cdot s$*' for '*$s \in E(\phi, \mathfrak{M})$*'.*
- *If* σ *is a sentence (that is to say* $\sigma \in CF$ *and* $\mathtt{VL}(\sigma) = \emptyset$*) we write '*$\mathfrak{M} \models \sigma$*' for* $\mathfrak{M} \models \sigma \cdot \emptyset$*.*
- *If* T *is a set of sentences in* CF *we write '*$\mathfrak{M} \models T$*' for* $(\forall \sigma \in T)(\mathfrak{M} \models \sigma)$*.*

'$\mathfrak{M} \models \phi \cdot s$' is a stratified [homogeneous] formula wherein all free variables receive the same type.

3.1.3 Properties of $\models$

Let $\mathfrak{M} = \langle A, R \rangle$ be a structure and ϕ a concrete formula of $\mathcal{L}(\in, =)$, the language of NF. When $s \in S(\ulcorner \phi \urcorner, \mathfrak{M})$ we can define the interpretation of ϕ in $\mathfrak{M}$ for the free variables given by s, which we write $\phi^{\mathfrak{M}_s}$—this being the formula obtained from ϕ by replacing '$\in$' by 'R', restricting all quantifiers to A and replacing each free variable x by $s(\ulcorner x \urcorner)$.

Proposition 39 *NF* $\vdash \mathfrak{M} \models \phi \cdot s \longleftrightarrow \phi^{\mathfrak{M}_s}$.

Suppose $\ulcorner\phi\urcorner \in CF$ and ϕ is of the form $\exists x_1 \exists x_2 \exists x_3 \ldots \exists x_n \psi$ where $\mathtt{VL}(\psi) = \{2i_1, 2i_2 \ldots 2i_n, 2j_1, 2j_2 \ldots 2j_k\}$. If now $\mathfrak{M} = \langle A, R\rangle$ we have

$$\mathrm{NF} \vdash \mathfrak{M} \models \ulcorner\phi\urcorner \cdot s \longleftrightarrow (\exists r)((r : \{2i_1 \ldots 2i_n\} \to A) \wedge (\mathfrak{M} \models \ulcorner\psi\urcorner \cdot (s \cup r)))$$

We prove this by induction on the length of formulæ.[3] Of course there is an analogous result for formulæ whose quantifier prefix is $\forall \ldots \forall$.

3.2 Interpretations in NF

Let T be a theory in $\mathcal{L}(\in, =)$, let A^* be a class of sets in NF and $R^* \subseteq A^* \times A^*$ be a "relation" on A^*. We say that $\mathfrak{M}^* = \langle A^*, R^*\rangle$. We say that $\mathfrak{M}^* = \langle A^*, R^*\rangle$ is an **interpretation** of T iff for every axiom σ of T one can prove $\sigma^{\mathfrak{M}^*}$—where $\sigma^{\mathfrak{M}^*}$ is the interpretation of σ in $\mathfrak{M}^*$.

If $\mathfrak{M}$ is a model of T in NF then $\mathfrak{M}$ is an interpretation of T in NF. The converse is well-known to be false.

3.3 Connections with Consistency

It is well-known that if one can show in a theory (as it might be an extension of NF) that there is a model of some other theory T then one has shown the consistency of T in the first theory. Let us remind ourselves that whenever we can interpret a theory T' in another theory T then we have established Con(T) $\to$ Con(T').

[3] I think he really means *by structural induction on formulæ* but let's not quibble.

Chapter 4

Models of Z

4.1 The Set Theory of Zermelo

[For the moment I am not supplying a translation of this section because it consists solely of a tabulation of the axioms of Zermelo, and we all know what they are do we not?!]

4.2 Interesting substructures of $\langle BF, \mathcal{E} \rangle$

4.2.1 Definitions

Let S be a well-founded extensional relation. We define S^α ('α' ranging over ordinals) by recursion:

$S^0 := \{a \in dom(S) : \neg(\exists b \in dom(S))(b \; S \; a)\}$;

$S^{\alpha+1} := \{a \in dom(S) : (\forall b)(b \; S \; a) \to b \in S^\alpha\}$;

$S^\lambda := \bigcup_{\alpha<\lambda} S^\alpha$ for λ limit.

We observe that '$x = S^\alpha$' is stratified [homogeneous].

We will often write '$\langle M, \mathcal{E} \rangle$' instead of '$\langle M, \mathcal{E} \restriction M \rangle$' when $M \subseteq BF$. We will write 'R^α' instead of '$\mathcal{E}^\alpha$' and we write 'M_α' for '$R^{\omega_0+\alpha}$'. Induction on the ordinals easily establishes that $\langle BF, \mathcal{E} \rangle$ is an end-extension of $\langle R^\alpha, \mathcal{E} \rangle$ and that $\alpha \mapsto R^\alpha$ is monotone increasing. Further, M_0 is precisely $\{\mathcal{T}(R) \in BF : |dom(R)| < \aleph_0\}$ and $|M_0| = \aleph_0$.

4.2.2 Rank

Definition 40 $BFR := \bigcup_{\alpha \in NO} M_\alpha$. *We have a rank function ρ for elements of BFR: $\rho(a)$ is the least α such that $a \in M_\alpha$.*

$\langle BF, \mathcal{E} \rangle$ is clearly an end-extension of $\langle BFR, \mathcal{E} \rangle$.

$M^* := \bigcup_{\alpha \in FC} M_\alpha$ ... where FC is the class of strongly cantorian ordinals. Evidently $\mathfrak{a} \in M^* \to \rho(\mathfrak{a}) \in FC$.

4.3 Models of Z

4.3.1 Definitions

Recall the definition of exponentiation from p.12. It has the consequence that $2^x \neq \emptyset \to (\exists y)(x = Ty)$. Let σ be the assertion that $\Phi(\aleph_0) \notin FIN$. σ is evidently equivalent to the assertion that the function $n \mapsto \beth(n)$ is defined on all of $\mathbb{N}$.[1]

Proposition 41 *$NF + \sigma \vdash (\forall n \in \mathbb{N})(\exists y \in NC)(\beth(n) = Ty)$.*

Proof: By σ we know that, for all $n \in \mathbb{N}$, $\beth(n)$ is defined. By the asterisked remark in the preceding paragraph we infer that $\beth(n)$ must be T of something. ■

Proposition 42 *$NF + \sigma \vdash (\forall n \in \mathbb{N})(\beth(Tn) = T\beth(n))$.*

Proof:

By an easy induction on $\mathbb{N}$.[2] True for $n = 0$, since $\aleph_0 = |\mathbb{N}|$, and $\mathbb{N}$ is cantorian. Now assume true for n; deduce it for $n+1$. It's easy to check in NF that $\mathcal{P}(\iota``x)$ and $\iota``(\mathcal{P}(x))$ are the same size [for $y \subseteq x$ swap $\iota``y$ with $\{y\}$]. Thus $2^{Ty} = T2^y$ whenever 2^y is defined, and we infer $\beth(Tn+1) = 2^{T\beth(n)} = T2^{\beth(n)} = T\beth(n+1)$ as desired. ■

Proposition 43 *For each concrete k, $NF + \sigma \vdash (\forall n \in \mathbb{N})(\beth(n) < T^k|V|)$.*

Proof: We do know at least that $(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(m = Tn)$. We prove this easily by induction [using the axiom of infinity]. From this it follows that, for each concrete k, $(\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(n = T^k m)$. So $\beth(n) = \beth(T^k m)$. So, by proposition 41, we deduce that $\beth(T^k m) = T^k(\beth(m))$. Since $(\forall m \in \mathbb{N})(\beth(m) < |V|)$ we infer $\beth(n) = T^k(\beth(m)) < T^k|V|$. ■

Proposition 44 *$NF + \sigma \vdash (\forall n \in \mathbb{N})(|M_n| \leq \beth_{Tn})$.*

Proof: '$|M_n| \leq \beth_{Tn}$' is stratified so we can prove it by induction [on 'n']. The fact that $|M_0| = \aleph_0$ is the last observation before section 4.2.2, and that takes care of the base case. For the induction step, consider $\mathfrak{a} \in M_{n+1}$; by definition

[1] In what immediately follows we are using the letter '$\beth$' to denote the function one beyond exponentiation in the hierarchy of increasing functions $\mathbb{N} \to \mathbb{N}$. Later we will use it in its more usual transfinite way. When we do that we will write the argument as a subscript in the approved fashion, not in argument place as we have done here.

[2] It's true for all cardinals, and proved directly, not by induction.

of the M_αs we infer $\{\mathfrak{b} : \mathfrak{b}\ \mathcal{E}\ \mathfrak{a}\} \subseteq M_n$. Evidently the map $g : \iota``M_{n+1} \to \mathcal{P}(M_n)$ defined by $\{\mathfrak{a}\} \mapsto \{\mathfrak{b} : \mathfrak{b}\ \mathcal{E}\ \mathfrak{a}\}$ is injective. This gives $T|M_{n+1}| \leq |\mathcal{P}(M_n)|$. $2^{T|M_n|} = |\mathcal{P}(M_n)|$ by definition. Therefore $T|M_{n+1}| \leq 2^{T|M_n|}$. By induction hypothesis $|M_n| \leq \beth_{Tn}$ so we have $T|M_{n+1}| \leq 2^{|M_n|} \leq 2^{T\beth_{Tn}} = T(2^{\beth_{Tn}})$. ■

Proposition 45 *$NF + \sigma \vdash (\forall \mathfrak{a} \in M_{\omega_0})(\exists \mathfrak{b} \in BF)(\mathfrak{a} = T^2\mathfrak{b})$.*

Proof: If $\mathfrak{a} \in M_{\omega_0}$ then $\mathfrak{a} \in M_n$ for some $n \in \mathbb{N}$. So $|dom(seg_{\mathcal{E}}(\mathfrak{a}))| \leq |M_n|$. By the two immediately preceding propositions we can infer $|(M_n)| \leq T^4|V|$. So $|dom(seg_{\mathcal{E}}(\mathfrak{a}))| < T^4|V|$. So there must be an injection from $dom(seg_{\mathcal{E}}(\mathfrak{a}))$ into $\iota^4``V$, and there must be a relation S with $seg_{\mathcal{E}}(\mathfrak{a}) \simeq RUSC^4(S)$. Now $\mathcal{T}(seg_{\mathcal{E}}(\mathfrak{a})) = T^2\mathfrak{a}$. If now $\mathfrak{b} := \mathcal{T}(S)$ we have $T^2\mathfrak{a} = \mathcal{T}(seg_{\mathcal{E}}(\mathfrak{a})) = T^4\mathfrak{b}$, whence $\mathfrak{a} = T^2\mathfrak{b}$. ■

Recall that Z$\backslash P$ is Z minus the axiom of power set.

Lemma 46 *$NF \vdash$ For all limit ordinals λ, $\langle M_\lambda, \mathcal{E}\rangle \models Z\backslash P$.*

Proof: We will deal with the axioms individually.

1. $\langle M_\lambda, \mathcal{E}\rangle \models$ Extensionality because $\mathcal{E}$ is extensional and BF is an end-extension of $\langle M_\lambda, \mathcal{E}\rangle$.

2. $\langle M_\lambda, \mathcal{E}\rangle \models$ Emptyset. $\mathcal{T}(\emptyset)$ is the empty set of this model.

3. Pairing. We have to check that $\langle M_\lambda, \mathcal{E}\rangle \models (\forall xy)(\exists z)(z = \{x, y\})$. That is to say, we want

$$(\forall \mathfrak{a}, \mathfrak{b} \in M_\lambda)(\exists \mathfrak{c} \in M_\lambda)(\forall \mathfrak{d} \in M_\lambda)(\mathfrak{d}\ \mathcal{E}\ \mathfrak{c} \longleftrightarrow \mathfrak{d} = \mathfrak{a} \vee \mathfrak{d} = \mathfrak{b})$$

Let $\mathfrak{a} = \mathcal{T}(R)$ and $\mathfrak{b} = \mathcal{T}(S)$ be two elements of M_λ. Now let
$R' := \{\langle\langle a, 0\rangle, \langle b, 0\rangle\rangle : \langle a, b\rangle \in R\}$ and
$S' := \{\langle\langle a, 1\rangle, \langle b, 1\rangle\rangle : \langle a, b\rangle \in S\}$.
Obviously $R \simeq R'$ and $S \simeq S'$ so $\mathfrak{a} = \mathcal{T}(R')$ and $\mathfrak{b} = \mathcal{T}(S')$ and $dom(R') \cap dom(S') = \emptyset$. Now let

$$A := \{z \in dom(S') : \neg(\exists t)(seg_{R'}(t) \simeq seg_{S'}(z))\}$$

Now let

$$W := R' \cup (S' \restriction A) \cup \{\langle u, v\rangle : v \in A \wedge u \in dom(R) \wedge (\exists t \in dom(S))(seg_{R'}(u) \simeq seg_{S'}(t) \wedge tS'v)\}$$

and we have $dom(W) = dom(R') \cup A$.

Now let us define $W' := W \cup \{\langle \mathbb{1}_R, \langle V, 2\rangle\rangle, \langle \mathbb{1}_S, \langle V, 2\rangle\rangle\}$.

$\mathfrak{a}$ and $\mathfrak{b}$ are in M_λ. Since λ is limit there is $\beta < \lambda$ with $\{\mathfrak{a}, \mathfrak{b}\} \subseteq M_\beta$. We check that $\mathcal{T}(W') \in M_{\beta+1}$ and that

$$(\forall \mathfrak{d} \in M_\lambda)(\mathfrak{d}\ \mathcal{E}\ \mathcal{T}(W') \longleftrightarrow \mathfrak{d} = \mathfrak{a} \vee \mathfrak{d} = \mathfrak{b})$$

... so that $\mathcal{T}(W')$ really is $\{\mathfrak{a}, \mathfrak{b}\}$ in the sense of $\langle M_\lambda, \mathcal{E}\rangle$. ■[3]

4. $\langle M_\lambda, \mathcal{E}\rangle \models$ Axiom of Sumset. Given $\mathfrak{a} \in M_\lambda$, with $\mathfrak{a} = \mathcal{T}(R)$, consider

$$B := \{b \in dom(R) : (\exists c \in dom(R))(b\ R\ c \wedge c\ R\ (\mathbb{1}_R))\}$$

and let ζ be something not in $dom(R)$. If $S = \bigcup\{seg_R(b) : b \in B\} \cup \{\langle b, \zeta\rangle : b \in B\}$ we check easily that $\mathcal{T}(S)$ is the sumset of $\mathfrak{a}$ in the sense of $\langle M_\lambda, \mathcal{E}\rangle$. ■

5. $\langle M_\lambda, \mathcal{E}\rangle \models$ Axiom of Infinity. We saw earlier (formula 2.1 page 22) how to find a homomorphic injection $f : NO \to BF$ (so $\alpha < \beta \longleftrightarrow (f(\alpha)\ \mathcal{E}\ f(\beta))$). Let us write $\overline{\alpha}$ for the encoding of the ordinal α as a member of BF. We will show that $\overline{\omega_0} \in M_\lambda$. It's easy to check that

$$\mathfrak{a}\ \mathcal{E}\ \overline{\omega_0} \longleftrightarrow (\exists n \in \mathbb{N})(\mathfrak{a} = \mathcal{T}(\{\langle m, k\rangle : m, k \in \mathbb{N} \wedge m < k \wedge k \leq n\})).$$

Now, for $\mathfrak{a}\ \mathcal{E}\ \overline{\omega_0}$ with[4] $\mathfrak{a} = \mathcal{T}(R)$, we must have $|dom(R)| < \aleph_0$. So $\mathfrak{a}\ \mathcal{E}\ \overline{\omega_0} \to \mathfrak{a} \in M_0$, whence $\overline{\omega_0} \in M_\lambda$. It is easy to show that $\overline{\omega_0}$ is the von Neumann ordinal ω in the sense of $\langle M_\lambda, \mathcal{E}\rangle$.[5] ■

6. $\langle M_\lambda, \mathcal{E}\rangle \models$ Axiom of Foundation. This follows easily from the fact that $\mathcal{E}$ is well-founded.

7. $\langle M_\lambda, \mathcal{E}\rangle \models$ the scheme of separation.[6] We will have to show that

NF $\vdash (\forall \phi \in CF)[\ulcorner y \urcorner \not\in VL(\phi) \to (\forall s \in S(\phi, M_\lambda))[(\langle M_\lambda, \mathcal{E}\rangle \models \ulcorner \forall x \exists y \forall t(t \in y \longleftrightarrow (t \in x \wedge \phi))\urcorner s$.

$$\langle M_\lambda, \mathcal{E}\rangle \models \ulcorner \forall x \exists y \forall t(t \in y \longleftrightarrow (t \in x \wedge \phi)]\urcorner s]$$

is equivalent to

$$(\forall \mathfrak{a} \in M_\lambda)(\exists \mathfrak{b} \in N_\lambda)(\forall \mathfrak{c} \in M_\lambda)[\mathfrak{c}\ \mathcal{E}\ \mathfrak{b} \longleftrightarrow (\mathfrak{c}\ \mathcal{E}\ \mathfrak{a} \wedge \langle M_\lambda, \mathcal{E}\rangle \models \phi(s \cup \{\langle \ulcorner t \urcorner, \mathfrak{c}\rangle\}))].$$

(We are assuming that the variable t is free in ϕ: if not there is nothing to do).

[3] There is a lot of fiddly detail here but the key point is not in the fiddly detail. What matters is that we can make copies of R and S that are **disjoint**. We are trying to make a *two*-membered set and to do this we need *two* relations with disjoint domains. If the set we are trying to prove the existence of has κ members we will need—at least if we are to procede by this construction—κ many relations whose domains are pairwise disjoint. The task: `on being given a family` $\mathcal{F}$ `of relations, find a pairwise disjoint family` $\mathcal{F}'$ `of copies of members of` $\mathcal{F}$ can be discharged only if $|\mathcal{F}| \leq T|V|$. This becomes a problem when we want to show that the upper reaches of BF satisfy Power set for example.

[4] The original has '$\in$' here not '$\mathcal{E}$' but I think that is a mistake.

[5] I don't think it's easy, but it is certainly true. I might supply a proof in later draughts.

[6] The key point here is that $\mathcal{E}$ is *homogeneous* as opposed to merely *stratified*. This means that the $\mathcal{E}$-version *of any formula whatever* is stratified.

Suppose $\mathfrak{a} \in M_\lambda$. Then $\mathfrak{a}$ is $\mathcal{T}(R)$ for some $R \in \Omega$. Consider now the set

$$B := \{\mathfrak{c}\ \mathcal{E}\ \mathfrak{a} : \langle M_\lambda, \mathcal{E}\rangle \models \phi(s \cup \{\langle \ulcorner t \urcorner, \mathfrak{c}\rangle\})\}.$$

We saw in subsection 3.1.2 that $\langle M_\lambda, \mathcal{E}\rangle \models \phi(s \cup \{\langle \ulcorner t \urcorner, \mathfrak{c}\rangle\})\}$ is a stratified formula. Accordingly NF proves that B is a set. Now let B' be $\{z \in dom(R) : \mathcal{T}(seg_R(z)) \in B\}$, and set $S := \bigcup\{seg_R(z) : z \in B'\} \cup \{\langle z, \zeta\rangle : z \in B'\}$ where $\zeta \in V \setminus dom(R)$. Finally, setting $\mathfrak{b} := \mathcal{T}(S)$, we verify that

$$(\forall \mathfrak{c} \in M_\lambda)[\mathfrak{c}\ \mathcal{E}\ \mathfrak{b} \longleftrightarrow (\mathfrak{c}\ \mathcal{E}\ \mathfrak{a} \wedge \langle M_\lambda, \zeta\rangle \models \phi(s \cup \{\langle \ulcorner t \urcorner, \mathfrak{c}\rangle\}))]$$

If $\mathfrak{a} \in M_\lambda$ then—since λ is limit—$\mathfrak{a} \in M_\beta$ for some $\beta < \lambda$. If $\mathfrak{c}\ \mathcal{E}\ \mathfrak{a}$ we have $\mathfrak{c} \in M_\beta$. So $\mathfrak{b} \in M_{\beta+1}$ and $\mathfrak{b} \in M_\lambda$.

Theorem 47 *NF + $\sigma \vdash$ $\langle M_\omega, \mathcal{E}\rangle \models$ Zermelo set theory.*

Proof: By the preceding lemma we know that $\langle M_\omega, \mathcal{E}\rangle \models$ all the axioms of Zermelo set theory except possibly Power set. Let $\mathfrak{a}$ be a member of M_ω we will show that it has a power set-in-the-sense-of-$\langle M_\omega, \mathcal{E}\rangle$. $\mathfrak{a} \in M_n$ for some $n \in \mathbb{N}$. Let B be the set $\{\mathfrak{b} \in BF : (\forall \mathfrak{c})(\mathfrak{c}\ \mathcal{E}\ \mathfrak{b} \to \mathfrak{c}\ \mathcal{E}\ \mathfrak{a})\}$. Clearly $B \subseteq M_n$. By proposition 45 we know that $(\forall \mathfrak{b} \in B)(\exists \mathfrak{b}' \in BF)(\mathfrak{b} = T^2\mathfrak{b}')$ and $\mathfrak{b}$ is of course $\mathcal{T}(seg_{\mathcal{E}}(\mathfrak{b}'))$. Now set $S := \bigcup\{seg_{\mathcal{E}}(\mathfrak{b}') : T^2\mathfrak{b}' \in B\} \cup \{\langle \mathfrak{b}', V\rangle : T^2\mathfrak{b}' \in B\}$, and $\mathfrak{d} := \mathcal{T}(S)$. We now verify that $\mathfrak{d} \in M_{n+1}$ and therefore $\mathfrak{d} \in M_\omega$. Finally we need to check that $\mathfrak{d}$ is also, as desired, the power set of $\mathfrak{a}$ in-the-sense-of-$\langle M_\omega, \mathcal{E}\rangle$. Evidently $(\forall \mathfrak{b} \in M_\omega)(\mathfrak{b}\ \mathcal{E}\ \mathfrak{d} \longleftrightarrow \mathfrak{b} \in B)$, which shows that $\langle M_\omega, \mathcal{E}\rangle \models \ulcorner b = \mathcal{P}(z)\urcorner[(z, \mathfrak{a}), (b, \mathfrak{d})]$ which was what we wanted.

Corollary 48

1. *NF + $\Phi(\aleph_0)$ is infinite $\vdash$ Con(Zermelo)*
2. *NF + AxCount$_\leq$ $\vdash$ Con(Zermelo)*[7]
3. *NF + the Axiom of Counting $\vdash$ Con(Zermelo).*

Proof:

1. Obvious ■

2. Assume AxCount$_\leq$; it will suffice to prove that $\beth_n$ is defined for all $n \in \mathbb{N}$. We observe:

$$\text{NF} \vdash (\forall n \in \mathbb{N})(\beth_n \text{ is defined} \to \beth_{Tn} \text{ is defined} \wedge T\beth_n = \beth_{Tn}) \qquad (4.1)$$

[7]Hinnion has not defined AxCount$_\leq$ but the omission is easily remedied: AxCount$_\leq$ states that $(\forall n \in \mathbb{N})(n \leq Tn)$. It was becoming clear at the time Hinnion was writing this thesis that many of the interesting consequences of the Axiom of Counting followed from this weaker version.

We prove the observation by induction on n. Clearly the case $n = 0$ is good: $\aleph_0 = T\aleph_0$.

For the induction step suppose we have ($\beth_n$ is defined) $\to T\beth_n = \beth_{Tn}$, and suppose $\beth_{n+1}$ is defined.

We want $T\beth_{n+1} = \beth_{Tn+1}$. Since $\beth_{n+1}$ is defined, so is $\beth_n$. By definition of the beth numbers we have

$$\beth_{Tn+1} = 2^{\beth_{Tn}} = 2^{T\beth_n} = T(\beth_{n+1})$$

This concludes the proof of the observation. [So far we have not used $\text{AxCount}_{\leq}$.] Next we prove by induction on n that $\beth_n$ is defined for all $n \in \mathbb{N}$. Clear for $n = 0$. For the induction step suppose $\beth_n$ is defined. By the observation we have $T\beth_n = \beth_{Tn}$, and by definition $\beth_{n+1} = 2^{\beth_n}$. Since $n \leq Tn$ [by $\text{AxCount}_{\leq}$] and $\beth_n$ and $\beth_{Tn}$ are defined we have $\beth_n \leq \beth_{Tn}$. So $\beth_n \leq T\beth_n$, and there is $y \in NC$ with $\beth_n = Ty$. Therefore $\beth_{n+1}$ is defined (since it is equal to 2^{Ty}). ■

3. NF + Axiom of Counting $\vdash$ Con(Zermelo). [The Axiom of Counting implies $\text{AxCount}_{\leq}$; then use part (2) above.]

In NF, the Axiom of Counting implies that $\Phi(\aleph_0)$ is infinite. The converse is not known, nor is it known if it is consistent relative to NF that $\Phi(\aleph_0)$ be infinite.[8] Henson [2] showed that the Axiom of Counting is not a theorem of any consistent stratified extension of NF, so it can't be a consequence of " $\Phi(\aleph_0)$ is infinite" unless that assertion is inconsistent.

4.4 Further remarks

The methods of this chapter can be used to prove the following analogues of lemma 46:

1. NF $\vdash \langle BF, \mathcal{E}\rangle \models$ Zermelo $\setminus$ Power set
2. NF $\vdash \langle BFR, \mathcal{E}\rangle \models$ Zermelo $\setminus$ Power set

[BFR is defined at the beginning of section 4.2.2 on page p. 33.]

Let TCl be the axiom of transitive closure: every set has a $\subseteq$-minimal transitive superset.[9] It is known that TCl is a theorem of ZF but not of Z unless Z is inconsistent. We claim the following:

Remark 49 [10]

1. *NF* $\vdash \langle BF, \mathcal{E}\rangle \models$ *TCl;*

[8] This remains as true in 2008 as it was in 1975.

[9] In the original Hinnion calls this axiom 'Cl'. The notation 'TCl' is more standard nowadays.

[10] This remark is not numbered in the original. I have given it a number so LaTeXcan cross-reference it.

2. *NF* $\vdash$ $\langle BFR, \mathcal{E}\rangle \models$ *TCl;*

3. *NF* $\vdash$ $(\forall \alpha \in NO)(\langle M_\alpha, \mathcal{E}\rangle \models$ *TCl*$)$.

We prove the last of these claims.

Let $\mathfrak{a} = \mathcal{T}(R)$ be a member of M_α, and define

$$S := (R \setminus \{\langle x, (\mathbb{1}_R)\rangle : x \; R \; (\mathbb{1}_R) \wedge x \in dom(R)\}) \cup \{\langle x, \zeta\rangle : x \in (dom(R) \setminus \{\mathbb{1}_R\})\}$$

... where $\zeta \notin dom(R)$. We verify that $\mathcal{T}(S)$ is the transitive closure of $\mathfrak{a}$ in the sense of $\langle M_\alpha, \mathcal{E}\rangle$. ■

As things stand we do not know whether or not the Axiom of Counting and AxCount$_\leq$ are equivalent.[11]

Specker has asked: can we prove that there are infinitely many distinct cardinals of infinite sets? [still open in 2010]. A related (harder) challenge is to prove in NF that $\Phi(\aleph_0)$ is infinite. [The general view nowadays is that this is not a theorem of NF unless NF is inconsistent.]

[11] It's still not clear that AxCount$_\leq$ is weaker than the axiom of counting!

Chapter 5

Models of ZF and ZF-minus-Power-Set in Extensions of NF

5.1 Zermelo-Fraenkel Set Theory

The axiom of ZF set theory are those of Zermelo set theory plus the scheme of **replacement**:

$$(\forall x)(\exists! y)\phi(x, y, \vec{w}) \to (\forall z)(\exists t)(\forall y)(y \in t \longleftrightarrow (\exists x \in z)\phi(x, y, \vec{w}))$$

It is standard that replacement subsumes separation.

5.2 A model of ZF

Let σ_1 be the sentence[1]

$$(\exists x \in NC) \bigwedge \left(\begin{array}{l} x > \aleph_0 \\ (\forall y \in NC)(y < x \to 2^y < x) \\ (\forall y \in NC)(Ty < x \longleftrightarrow y < x) \\ (\forall b)((|b| < x \wedge (\forall a \in b)(|a| < x)) \to |\bigcup b| < x) \end{array} \right) \qquad (\sigma_1)$$

Remarks:

- If $x = Tx$ then x satisfies the third condition.
- If $x \leq Tx \vee Tx \leq x$ the third condition simplifies to $x = Tx$.

[1]In the original text the second clause has the additional stipulation that 2^y be defined. I've left it out because if $y < x$ then the fact that 2^y is defined follows from the third clause. It's also a pain typesetting Roman inside a formula!!

- These conditions closely resemble the conditions familiar from ZF for a cardinal to be strongly inaccessible. [2]

Theorem 50 *NF + $\sigma_1 \vdash$ Con(ZF).*

Proof: Let x be a cardinal of the flavour whose existence σ_1 alleges, and let M be $\{\mathfrak{a} \in BF : |dom(seg_{\mathcal{E}}(\mathfrak{a}))| < x\}$. It will come as no surprise to the reader that $\langle M, \mathcal{E}\rangle$ is a model of ZF.

1. $\langle M, \mathcal{E}\rangle$ is an initial sgment of $\langle BF, \mathcal{E}\rangle$. [That is to say, $\langle BF, \mathcal{E}\rangle$ is an end-extension of $\langle M, \mathcal{E}\rangle$ and so $\langle M, \mathcal{E}\rangle$ is a substructure of $\langle BF, \mathcal{E}\rangle$ that is transitive-in-the-sense-of- $\mathcal{E}$.]

2. An easy modification of the proof of lemma 46 of ch 4 shows that $\langle M, \mathcal{E}\rangle$ is a model of Zermelo \ Power set.

3. For $k = 0, 1, 2, \ldots$ we have $y < x \to y < T^k|V|$. So $y < x \to 2^y$ is defined and that y is Tz for some cardinal z. From $Tz < x$ we can infer (by the third clause in σ_1) that $z < x < |V|$ whence $y < Tx < T|V|$. Since $z < x$ there is a cardinal z' such that $z = Tz'$. In the same way we infer that $z < T|V|$ and that $y < T^2|V|$, and $y < T^k|V|$ for each concrete natural k.

4. We want $(\forall \mathfrak{a} \in M)(\exists \mathfrak{b} \in BF)(\mathfrak{a} = T^2\mathfrak{b})$. For $\mathfrak{a} \in M$, $|(dom(seg_{\mathcal{E}}(\mathfrak{a}))| < x$. By point 3 above we infer $|dom(seg_{\mathcal{E}}(\mathfrak{a}))| < T^4|V|$. Therefore there is a relation S such that $seg_{\mathcal{E}}(\mathfrak{a}) \simeq RUSC^4(S)$, and accordingly $T^2\mathfrak{a} = \mathcal{T}(seg_{\mathcal{E}}(\mathfrak{a})) = T^4\mathcal{T}(S)$. $\mathfrak{a}$ is now $T^2\mathcal{T}(S)$, so the $\mathfrak{b}$ we seek is $\mathcal{T}(S)$.

5. $\langle M, \mathcal{E}\rangle \models$ Power set. Given $\mathfrak{a} \in M$ set $B_{\mathfrak{a}} := \{\mathfrak{b} \in BF : \forall \mathfrak{c}(\mathfrak{c}\ \mathcal{E}\ \mathfrak{b} \to \mathfrak{c}\ \mathcal{E}\ \mathfrak{a})\}$. Evidently whenever $\mathfrak{b} \in B$ we have $|dom(seg_{\mathcal{E}}(\mathfrak{b}))| \leq |dom(seg_{\mathcal{E}}(\mathfrak{a}))|$, so $B_{\mathfrak{a}} \subseteq M$. Now by point 4 we have $(\forall \mathfrak{b} \in B)(\exists \mathfrak{b}' \in BF)(\mathfrak{b} = T^2\mathfrak{b}')$, so let $B'_{\mathfrak{a}} := \{\mathfrak{b}' \in BF : T^2\mathfrak{b}' \in B_{\mathfrak{a}}\}$, and
$S := \bigcup\{seg_{\mathcal{E}}(\mathfrak{b}') : \mathfrak{b}' \in B'_{\mathfrak{a}} : \mathfrak{b}' \in B'_{\mathfrak{a}}\} \cup \{\langle \mathfrak{b}', V\rangle : \mathfrak{b}' \in B'_{\mathfrak{a}}\}$.
Let $\mathfrak{d}$ be $\mathcal{T}(S)$. We will show that $\mathfrak{d} \in M$. We have $dom(seg_{\mathcal{E}}(\mathfrak{d})) = \bigcup\{dom(seg_{\mathcal{E}}(\mathfrak{b})) : \mathfrak{b} \in B\} \cup \{\mathfrak{d}\}$, whence $|dom(seg_{\mathcal{E}}(\mathfrak{d}))| = |\bigcup\{dom(seg_{\mathcal{E}}(\mathfrak{b})) : \mathfrak{b} \in B\}|$. Now consider the set $C = \{dom(seg_{\mathcal{E}}(\mathfrak{b})) : \mathfrak{b} \in B\}$. We can see that $|C| = T|B|$. (The desired bijection between C and $\iota``B$ is the function $dom(seg_{\mathcal{E}}(\mathfrak{b})) \mapsto \{\mathfrak{b}\}$.) Next consider the map $g : \iota``B \to \mathcal{P}(\{\mathfrak{c} : \mathfrak{c}\ \mathcal{E}\ \mathfrak{a}\})$ given by $g(\{\mathfrak{b}\}) := \{z : z\ \mathcal{E}\ \mathfrak{b}\}$. Now $\mathcal{E}$ is extensional, so g is injective. This tells us that $|\iota``B| \leq |\mathcal{P}(\{\mathfrak{c} : \mathfrak{c}\ \mathcal{E}\ \mathfrak{a}\})|$. Also $\{\mathfrak{c} : \mathfrak{c}\ \mathcal{E}\ \mathfrak{a}\} \subseteq dom(seg_{\mathcal{E}}(\mathfrak{a}))$. Therefore $|\iota``B| < x$, and we conclude $|C| = T|B| < x$. Since additionally $(\forall c \in C)(|c| < x)$ we infer—from property 4 of x—that $|\bigcup C| < x$, which is to say that $|dom(seg_E(\mathfrak{d}))| < x$. So we have shown that $\mathfrak{d}$ is an element of M. We now verify easily that $\langle M, \mathcal{E}\rangle \models \ulcorner v = \mathcal{P}(u)\urcorner[\langle u, \mathfrak{a}\rangle, \langle v, \mathfrak{d}\rangle]$.

[2] But observe that x is not assumed to be an aleph. So the proof doesn't trade on the bog-standard fact that $V_\kappa \models$ ZF when κ is strongly inaccessible. Rather it uses H_κ (the set of sets hereditarily of size less than κ)—which of course is the same thing when κ is strongly inaccessible.

6. It now only remains to show that $\langle M,\ \mathcal{E}\,\rangle \models$ replacement.

   Let $\ulcorner\phi(u, v, x_1, x_2 \ldots x_n)\urcorner$ be a formula in CF such that $\langle M,\ \mathcal{E}\,\rangle \models \ulcorner(\forall x_1 \ldots x_n)(\forall u)(\exists! v)\phi(u, v, x_1, x_2 \ldots x_n)\urcorner$.

   We claim that $\langle M,\ \mathcal{E}\,\rangle \models \ulcorner(\forall x_1 \ldots x_n \forall z)(\exists t)(\forall v)(v \in t \longleftrightarrow (\exists u \in z)\phi(u, v, x_1, x_2 \ldots x_n))\urcorner$.

   This is equivalent to:

   For all s from $VL(\ulcorner\phi\urcorner)$ (which is to say, from $\{\ulcorner u\urcorner, \ulcorner v\urcorner\}$) to M,

   $$(\forall \mathfrak{c} \in M)(\exists \mathfrak{d} \in M)(\forall \mathfrak{b} \in M)[\mathfrak{b}\ \mathcal{E}\ \mathfrak{d} \longleftrightarrow$$

   $$\exists \mathfrak{a} \in M(\mathfrak{a}\ \mathcal{E}\ \mathfrak{c}\ \wedge\ \langle M, \mathcal{E}\rangle \models \ulcorner\phi(u, v, \cdots)\urcorner s \cup \{\langle \ulcorner u\urcorner, \mathfrak{a}\rangle, \langle \ulcorner v\urcorner, \mathfrak{b}\rangle\})]$$

   Now let s be a function from $VL(\ulcorner\phi\urcorner)$ (which is to say, from $\{\ulcorner u\urcorner, \ulcorner v\urcorner\}$) to M, and let $\mathfrak{c}$ be an element of M. Set

   $$g := \{\langle \mathfrak{a}, \mathfrak{b}\rangle : \langle M,\ \mathcal{E}\,\rangle \models \ulcorner\phi\urcorner(s \cup \{\langle \ulcorner u\urcorner, \mathfrak{a}\rangle, \langle \ulcorner v\urcorner, \mathfrak{b}\rangle\})\}$$

   It is clear that g maps M into itself. Now set

   $$B := \{g(\mathfrak{a}) : \mathfrak{a}\ \mathcal{E}\ \mathfrak{c}\}; \qquad A := \{\mathfrak{a} : \mathfrak{a}\ \mathcal{E}\ \mathfrak{c}\}$$

   Clearly g is a surjection $A \twoheadrightarrow B$, whence we infer $T|B| \leq |\mathcal{P}(A)|$. (Think of $h : \iota``B \to \mathcal{P}(A)$ given by $h(\{\mathfrak{b}\}) = \{\mathfrak{a} : g(\mathfrak{a}) = \mathfrak{b}\}$.)

   Since $A \subseteq dom(seg_{\mathcal{E}}(\mathfrak{c}))$, we have $|A| < x$. By property 3 of x, $|A| < x$ implies $T|A| < x$. By property 2 of x that in turn implies that $2^{T|A|} = |\mathcal{P}(A)| < x$. We conclude that $T|B| < x$ and $|B| < x$. Since g sends M into M we infer (from point 4 (above) in this proof that $(\forall \mathfrak{a} \in A)(\exists \mathfrak{b} \in BF)(g(\mathfrak{a}) = T^2\mathfrak{b})$. Consider now the set

   $$S := \bigcup\{seg_{\mathcal{E}}(\mathfrak{e}) : T^2\mathfrak{e} \in B\} \cup \{\langle \mathfrak{e}, V\rangle : T^2\mathfrak{e} \in B\}.$$

   Let $\mathfrak{d}$ be $\mathcal{T}(S)$; we will establish that $\mathfrak{d} \in M$. Evidently $dom(seg_{\mathcal{E}}(\mathfrak{d})) = \bigcup\{dom(seg_{\mathcal{E}}(\mathfrak{b})) : \mathfrak{b} \in B\} \cup \{\mathfrak{d}\}$. We have $|\{dom(seg_{\mathcal{E}}(\mathfrak{b})) : \mathfrak{b} \in B\}| = T|B| < x$, whence $|\{dom(seg_{\mathcal{E}}(\mathfrak{b})) : \mathfrak{b} \in B\}| = T|B| < x$. Since, furthermore, $|dom(seg_{\mathcal{E}}(\mathfrak{b}))| < x$ holds whenever $\mathfrak{b} \in B$, property 4 for x enables us to infer that $|\bigcup\{dom(seg_{\mathcal{E}}(\mathfrak{b})) : \mathfrak{b} \in B\}| < x$, whence $\mathfrak{d} \in M$.

   We verify easily that $(\forall \mathfrak{b} \in M)(\mathfrak{b}\ \mathcal{E}\ \mathfrak{d} \longleftrightarrow \mathfrak{b} \in B)$ which is exactly what was claimed.

5.3 A model of ZF \ Power set

σ_0 is the axiom that says that a union of countably many countable sets is countable:

$$(\forall x)((|x| \leq \aleph_0 \wedge (\forall y \in x)(|y| \leq \aleph_0)) \to |\bigcup x| \leq \aleph_0) \qquad (\sigma_0)$$

Theorem 51 *NF + $\sigma_0 \vdash$ Con(ZF $\setminus$ Power set)*

Proof:

Let M be $\{\mathfrak{a} \in BF : |dom(seg_{\mathcal{E}}(\mathfrak{a}))| \leq \aleph_0\}$. It is clear that $\langle M, \mathcal{E}\rangle$ is a transitive substructure of $\langle BF, \mathcal{E}\rangle$. We will need the following

Lemma 52 *NF* $\vdash (\forall \mathfrak{a} \in M)(\exists \mathfrak{b} \in BF)(\mathfrak{a} = T^2\mathfrak{b})$.

Proof: [3] We have $T\aleph_0 = \aleph_0$. Now let $\mathfrak{a}$ be an element of M, so $|dom(seg_{\mathcal{E}}(\mathfrak{a}))| \leq \aleph_0$. So, clearly, $|dom(seg_{\mathcal{E}}(\mathfrak{a}))| \leq T^4\aleph_0$. From this it follows that there is a relation S such that $seg_{\mathcal{E}}(\mathfrak{a}) \simeq RUSC^4(S)$, and therefore $\mathcal{T}(seg_{\mathcal{E}}(\mathfrak{a})) = T^4(\mathcal{T}(S))$. $\mathcal{T}(seg_{\mathcal{E}}(\mathfrak{a})) = T^2\mathfrak{a}$ always, so, if now $\mathcal{T}(S) = \mathfrak{b}$, we have $\mathfrak{a} = T^2\mathfrak{b}$. ∎

Now we return to the proof of the theorem and show specifically that NF + σ_0 proves that $\langle M, \mathcal{E}\rangle \models$ ZF $\setminus$ power set.

Extensionality

Easy: $\langle M, \mathcal{E}\rangle$ is a transitive substructure of $\langle BF, \mathcal{E}\rangle$ and $\mathcal{E}$ is extensional.

Axiom of Empty set

$\mathcal{T}(\emptyset)$ is the empty set of $\langle M, \mathcal{E}\rangle$.

Axiom of Pairing

Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are two elements of of M. By lemma 52 there are $\mathfrak{c}$ and $\mathfrak{d}$ in BF with $\mathfrak{a} = T^2\mathfrak{c}$ and $\mathfrak{b} = T^2\mathfrak{d}$. Therefore $\mathfrak{a} = \mathcal{T}(seg_{\mathcal{E}}(\mathfrak{c}))$ and $\mathfrak{b} = \mathcal{T}(seg_{\mathcal{E}}(\mathfrak{d}))$. Now let S be $seg_{\mathcal{E}}(\mathfrak{c}) \cup seg_{\mathcal{E}}(\mathfrak{d}) \cup \{\langle \mathfrak{c}, V\rangle, \langle \mathfrak{d}, V\rangle\}$; it is easy to check that $(\forall v \in M)(v \in \mathcal{T}(S) \longleftrightarrow (v = \mathfrak{a} \vee v = \mathfrak{b}))$. Furthermore $\mathcal{T}(S) \in M$—because $dom(seg_{\mathcal{E}}(\mathcal{T}(S))) = seg_{\mathcal{E}}(\mathfrak{a}) \cup seg_{\mathcal{E}}(\mathfrak{b}) \cup \{\mathcal{T}(S)\}$ and so is a union of three countable sets and is therefore countable. So $\mathcal{T}(S) \in M$ and is the witness we desire.[4]

Axiom of Sumset

Given $\mathfrak{a} \in M$ we can find $\mathfrak{b} \in BF$ with $\mathfrak{a} = \mathcal{T}(seg_{\mathcal{E}}(\mathfrak{b}))$ for the usual reasons; let C be $\{\mathfrak{c} \in dom(seg_{\mathcal{E}}(\mathfrak{b})) : (\exists \mathfrak{d})(\mathfrak{c}\ \mathcal{E}\ \mathfrak{d} \wedge \mathfrak{d}\ \mathcal{E}\ \mathfrak{b})\}$. If now $S = \bigcup\{seg_{\mathcal{E}}(\mathfrak{c}) : \mathfrak{c} \in C\} \cup \{\langle \mathfrak{c}, V\rangle : \mathfrak{c} \in C\}$ and $\mathfrak{d} = \mathcal{T}(S)$ we can easily check that $\mathfrak{d} \in M$ and that $\langle M, \mathcal{E}\rangle \models \ulcorner y = \bigcup x \urcorner[\langle x, \mathfrak{a}\rangle, \langle y, \mathfrak{d}\rangle]$.

Axiom of Infinity

Recall (p. 36) that $\overline{\alpha}$ is the element of BF corresponding to the ordinal α. Clearly $\overline{\omega_0} \in M$.

[3] Surely he means 'M' not 'BF' here?

[4] Hinnion appeals to countable choice here but it's only force of habit; countable choice is not needed to show that a union of three countable sets is countable.

Axiom of Foundation

This holds trivially because $\mathcal{E}$ is a well-founded relation.

Axiom Scheme of Replacement

Suppose $\langle M, \mathcal{E}\rangle \models \ulcorner \phi(x, y, x_1, x_2, \cdots x_n)\urcorner$, and let s be a function from $VL(\ulcorner \phi \urcorner) \setminus \{\ulcorner x \urcorner, \ulcorner y \urcorner\}$ to M. Let $\mathfrak{z}$ be an element of M. We claim:

$$(\exists \mathfrak{t} \in M)(\forall \mathfrak{b} \in M)(\mathfrak{b}\ \mathcal{E}\ \mathfrak{t} \longleftrightarrow (\exists \mathfrak{a}\ \mathcal{E}\ \mathfrak{z})(\langle M,\ \mathcal{E}\ \rangle \models \ulcorner \phi \urcorner s \cup \{\langle \ulcorner x \urcorner \mathfrak{a}\rangle, \langle \ulcorner y \urcorner, \mathfrak{b}\rangle\}))$$

Let g be $\{\langle \mathfrak{a}, \mathfrak{b}\rangle : \mathfrak{a}, \mathfrak{b} \in M \wedge \langle M, \mathcal{E}\rangle \models \ulcorner \phi \urcorner s \cup \{\langle \ulcorner x \urcorner \mathfrak{a}\rangle, \langle \ulcorner y \urcorner, \mathfrak{b}\rangle\}\}$. It is clear that g sends M into itself. Next set $B := \{g(\mathfrak{a}) : \mathfrak{a}\ \mathcal{E}\ \mathfrak{z}\}$. B is countable because $\mathfrak{z} \in M$. The lemma now tells us that $(\forall \mathfrak{b} \in B)(\exists \mathfrak{b}')(\mathfrak{b} = T^2\mathfrak{b}')$. Now let B' be $\{\mathfrak{b}' : T^2\mathfrak{b}' \in B\}$ and consider the relation $S := \bigcup\{seg_{\mathcal{E}}(\mathfrak{b}') : \mathfrak{b}' \in B'\} \cup \{\langle \mathfrak{b}', V\rangle : \mathfrak{b}' \in B'\}$. We will show that $\mathfrak{t} = \mathcal{T}(S)$ and $\mathfrak{t} \in M$. Which bloody lemma?

It is clear that $dom(seg_{\mathcal{E}}(\mathfrak{t})) = \bigcup\{dom(seg_{\mathcal{E}}(\mathfrak{b})) : \mathfrak{b} \in B\} \cup \{\mathfrak{t}\}$, whence $|dom(seg_{\mathcal{E}}(\mathfrak{t}))| = |\bigcup\{dom(seg_{\mathcal{E}}(\mathfrak{b})) : \mathfrak{b} \in B\}|$. Now let $A := \{dom(seg_{\mathcal{E}}(\mathfrak{b})) : \mathfrak{b} \in B\}$; we have $(\forall \mathfrak{a} \in A)(|\mathfrak{a}| \leq \aleph_0)$. Furthermore—in virtue of the function $\{\mathfrak{b}\} \mapsto dom(seg_{\mathcal{E}}(\mathfrak{b}))$—we have $|A| = T|B|$. $|B| \leq \aleph_0$ so $T|B| \leq T\aleph_0 = \aleph_0$. From $|A| \leq \aleph_0$ and $(\forall \mathfrak{a} \in A)(|\mathfrak{a}| \leq \aleph_0)$ and[5] axiom σ_0 we can deduce that $|\bigcup A| \leq \aleph_0$. It follows thence that $|dom(seg_{\mathcal{E}}(\mathfrak{t}))| \leq \aleph_0$, and that therefore $\mathfrak{t} \in M$.

It is now simple to check that

$$(\forall \mathfrak{b} \in M)(\mathfrak{b}\ \mathcal{E}\ \mathfrak{t} \longleftrightarrow (\exists \mathfrak{a}\ \mathcal{E}\ \mathfrak{z})(g(\mathfrak{a}) = \mathfrak{b}))$$

which is exactly what was claimed. ■

Since separation follows from replacement we have verified separation as well. This completes the proof that $\langle M, \mathcal{E}\rangle$ is a model of ZF\ power set.

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[5] The author writes '$Nc(\mathfrak{a})$' (which is Rosser-speak (from [4]) for '$|\mathfrak{a}|$') but I think this is an abuse of notation: I think he must mean *the cardinality of the carrier set of a typical representative of* $\mathfrak{a}$. This is something for which there is no convenient notation—hence the abuse.

Chapter 6

Interpreting Mac Lane set theory in NF

6.1 Mac Lane Set Theory

Δ_0 is the set of bounded formulæ (no unrestricted quantifiers) and Σ_1 formulæ are formulæ of the form $\exists x_1 \cdots x_n \phi$ where ϕ is Δ_0.[1]

6.2 Interpreting Mac + Σ_1-replacement into NF

Let M^* be the class

$$\bigcup_{\alpha \in FC} M_\alpha$$

Theorem 53 *$\langle M^*, \mathcal{E} \rangle$ gives an interpretation in NF of Mac Lane set theory + Σ_1-replacement + TCl.*

Proof:

1. It is easy to show that $\langle M^*, \mathcal{E} \rangle$ models ZF $\setminus$ power set and separation. The proof is just like that of lemma 46 of chapter 4. Unfortunately that method will not show that $\langle M^*, \mathcal{E} \rangle$ is a model of separation. This is because M^* is a class defined by an unstratified formula. This means that, for any formula ϕ, the interpretation ϕ^{M^*} of ϕ in $\langle M^*, \mathcal{E} \rangle$ is unstratified [even if ϕ itself is stratified] because of the restriction of the quantifiers to $\langle M^*, \mathcal{E} \rangle$. This has the effect that $\{\mathfrak{b} \in \mathfrak{a} : \phi^{M^*}\}$ is not necessarily going to be a set. Get this reference right

[1]Set theory is the result of restricting Zermelo set theory to Δ_0-separation. The theory Hinnion is interested in is Mac Lane plus Σ_1-replacement. He calls this theory '$Z_{\Delta_0} F_{\Sigma_1}$' but in the years since this work was written Mac Lane set theory has arrived on the scene and brings with it the more felicitous notation of "Mac + Σ_1-replacement" and that is the notation I shall use in this translation. The original title of this section was "The theory $Z_{\Delta_0} F_{\Sigma_1}$" For a detailed treatment of Mac Lane set theory see [16].

2. NF $\vdash (\forall \mathfrak{a} \in M^*)(\exists \mathfrak{b} \in M^*)(\mathfrak{a} = T^2\mathfrak{b})$. This is stratified and can be proved by induction on the ordinals in FC.

   Suppose it's true for α. Let $\mathfrak{z}$ be an element of $M_{T^2\alpha+1}$. Then $\mathfrak{a}\ \mathcal{E}\ \mathfrak{z} \to \mathfrak{a} \in M_{T^2\alpha}$. Then by induction hypothesis we have $\mathfrak{a} \in \mathfrak{z} \to (\exists \mathfrak{b} \in M_\alpha)(\mathfrak{a} = T^2\mathfrak{b})$. Set $A := \{\mathfrak{a} : \mathfrak{a}\ \mathcal{E}\ \mathfrak{z}\}$ and $B := \{\mathfrak{b} \in M_\alpha : T^2\mathfrak{b} \in A\}$. Since $\alpha = T^2\alpha$ we have $(\forall \mathfrak{b} \in M_\alpha)(\exists \mathfrak{c} \in M_\alpha)(\mathfrak{b} = T^2\mathfrak{c})$. We define
   $C := \{\mathfrak{c} \in M_\alpha : T^2\mathfrak{c} \in B\}$ and $S = \bigcup\{seg_{\mathcal{E}}(\mathfrak{c}) : \mathfrak{c} \in C\} \cup \langle \mathfrak{c}, V\rangle : \mathfrak{c} \in C\}$.
   If, finally, $\mathfrak{t} = \mathcal{T}(S)$ we find that $\mathfrak{t} \in M_{\alpha+1}$ and that $\mathfrak{z} = \mathcal{T}(seg_{\mathcal{E}}(\mathfrak{t}))$, whence $\mathfrak{z} = T^2\mathfrak{b}$.

   [That was the successor case; now for the limit case.]

   Let γ be a strongly cantorian limit ordinal with the proposition true for ordinals below γ. If $\mathfrak{a} \in M_\gamma$ then there is $\alpha < \gamma$ such that $\mathfrak{a} \in M_\alpha$. By induction hypothesis there is $\mathfrak{b} \in M_\beta$ with $\mathfrak{a} = T^2\mathfrak{b}$.

3. $\langle M^*, \mathcal{E}\rangle$ is a model of power set.

   If $\mathfrak{a} \in M^*$, then $(\exists \alpha \in FC)(\mathfrak{a} \in M_\alpha)$. Set $B := \{\mathfrak{b} \in BF : (\forall \mathfrak{c})(c\ \mathcal{E}\ \mathfrak{b} \to \mathfrak{c}\ \mathcal{E}\ \mathfrak{a})\}$ and $B' := \{\mathfrak{b} \in BF : T^2\mathfrak{b} \in B\}$. If $S = \bigcup\{seg_{\mathcal{E}}(\mathfrak{b}) : \mathfrak{b} \in B\} \cup \{\langle \mathfrak{b}, V\rangle : \mathfrak{b} \in B'\}$ we verify easily that $y = \mathcal{T}(S)$ is an element of $M_{\alpha+1}$ and that y is the power set of $\mathfrak{a}$ in the sense of the model.

4. $\langle M^*, \mathcal{E}\rangle$ is a model of Δ_0-separation. Suppose $\phi(y, x_1, x_2 \ldots x_n)$ is a Δ_0 formula. Then ϕ_{M^*}—which is its relativisation to $\langle M^*, \mathcal{E}\rangle$—is a *stratified* formula. If now $\mathfrak{a}, x_1 \ldots x_n$ are elements of M^* then $(\exists \alpha \in FC)(\{\mathfrak{a}, x_1 \ldots x_n\} \subseteq M_\alpha)$. Set $B := \{\mathfrak{b} \in M_\alpha : \mathfrak{b}\ \mathcal{E}\ \mathfrak{a} \wedge \phi_{M^*}(\mathfrak{b}, x_1 \ldots x_n)\}$. B is defined by a stratified set abstract and is therefore a set, as is $B' := \{\mathfrak{b} : T^2\mathfrak{b} \in B\}$. Now set $S := \bigcup\{seg_{\mathcal{E}}(\mathfrak{b}) : \mathfrak{b} \in B'\} \cup \{\langle \mathfrak{b}, V\rangle : \mathfrak{b} \in B'\}$ and $\mathfrak{c} := \mathcal{T}(S)$. Then (using item 2 above) we check that $\mathfrak{c} \in M_\alpha$ and

$$(\forall \mathfrak{b})[\mathfrak{b}\ \mathcal{E}\ \mathfrak{c} \longleftrightarrow (\mathfrak{b}\ \mathcal{E}\ \mathfrak{c} \wedge \phi_{M^*}(\mathfrak{b}, x_1 \ldots x_n))].$$

5. $\langle M^*, \mathcal{E}\rangle$ is a model of Σ_1-replacement.

   Suppose $\phi(x, y, x_1 \ldots x_n)$ is a Σ_1 formula and that $(\forall x_1 \ldots x_n)(\forall x)(\exists! y)(\phi_{M^*})$. In full ϕ looks like $(\exists y_1 \ldots y_k)\psi(x, y, x_1 \ldots x_n)$.

   For finite $A \subseteq BFR$ we can define the rank $\rho(A)$ of A by $\rho(A) :=$ least ordinal α such that $A \subseteq M_\alpha$. Now suppose $\mathfrak{a}, x_1 \ldots x_n$ are in M^*.

   Claim:

$$(\exists \mathfrak{b} \in M^*)(\forall y)[y\ \mathcal{E}\ \mathfrak{b} \longleftrightarrow (\exists x)(x\ \mathcal{E}\ \mathfrak{a} \wedge \phi_{M^*}(x, y, x_1 \ldots x_n))]$$

   Let $B := \{\rho(\{y_1, y_2 \ldots y_k\}) : \psi_{M^*}(x, y, y_1 \ldots y_k, x_1 \ldots x_n) \wedge x\ \mathcal{E}\ \mathfrak{a}\}$.

   B is defined by a stratified set abstract and is therefore a set. Since $(\forall x \in M^*)(\exists! y \in M)(\exists y_1 \ldots y_k \in M^*)\psi_{M^*}$ we infer $B \subseteq FC$, and thence $(\exists \alpha_0 \in FC)(\forall \beta \in B)(\beta < \alpha_0)$. (In effect, if $(\forall \alpha \in FC)(\exists \beta \in B)(\alpha \leq \beta)$

were true we could infer $FC = \{\alpha \in NO : (\exists \beta \in B)(\alpha \leq \beta)\}$, which would make FC a set; FC is known to be a proper class.)

We verify easily that

$$(\forall x)[x \mathcal{E} \mathfrak{a} \longleftrightarrow [(\exists y)\phi_{M^*} \longleftrightarrow (\exists y)\phi)_{M^*}]]$$

whence the class $W := \{y \in M^* : (\exists x)(x \mathcal{E} \mathfrak{a} \wedge \phi_{M^*})\}$ can be written as $\{y \in M_{\alpha_0} : (\exists x)(x \mathcal{E} \mathfrak{a} \wedge \phi_{M^*})\}$. ϕ_{M^*} is stratified so W is a set. So if $S := \bigcup\{seg_{\mathcal{E}}(y) : T^2 y \in W\} \cup \{\langle y, V\rangle : T^2 y \in W\}$ and $\mathfrak{b} := \mathcal{T}(S)$ we verify (using item 2 above) that $\mathfrak{b} \in M_{\alpha_0+1}$ and that

$$(\forall y)[y \mathcal{E} \mathfrak{b} \longleftrightarrow (\exists x)(x \mathcal{E} \mathfrak{a} \wedge \phi_{M^*}(x, y, x_1 \ldots x_n))]$$

as claimed.

6. We verify easily that $\langle M^*, \mathcal{E} \rangle$ is a model of TCl. The proof is analogous to that of remark 49 in chapter 4.

■

Corollary 54 *Con(NF) $\rightarrow$ Con(Mac + Σ_1 replacement)*

Chapter 7

Models in NF for fragments of Zermelo Set Theory

We saw above (chapter 4, theorem 5.3 p. 44) that there are models of ZF $\setminus$ power set in NF. Sadly, all attempts to prove the consistency of Zermelo set theory in NF have so far failed. In this chapter we obtain some positive results by weakening the power set axiom.

7.1 Lemmas provable in NF

(A) $\langle BF, \mathcal{E}\rangle \models \ulcorner x$ is a von Neumann ordinal$\urcorner[\langle x, a\rangle] \rightarrow \{\mathfrak{b} : \mathfrak{b}\ \mathcal{E}\ \mathfrak{a}\}$ is well-ordered by $\mathcal{E}$.

(B) If $\overline{\alpha}$ is the element of BF associated with the ordinal α (as on page 36) we prove by induction on α that

$$(\forall \mathfrak{a} \in BF)(\forall \alpha \in NO)[\alpha \geq \omega \rightarrow (\langle R_{T\alpha+1}, \mathcal{E}\rangle \models \ulcorner x \text{ is a von Neumann ordinal}\urcorner[\langle x, \mathfrak{a}\rangle] \longleftrightarrow (\exists \beta \leq \alpha)(\mathfrak{a} = \overline{\alpha})]$$

This shows that, in the model $\langle R_{T\alpha+1}, \mathcal{E}\rangle$, the von Neumann ordinals are objects of the form $\overline{\beta}$ with $\beta \leq \alpha$.

(C) For $k = 0, 1, 2, 3\ldots$ $\langle M_{\omega_k}, \mathcal{E}\rangle \models \ulcorner x$ is a von Neumann ordinal$\urcorner[\langle x, \mathfrak{a}\rangle] \rightarrow |\{\mathfrak{b} : \mathfrak{b}\ \mathcal{E}\ \mathfrak{a}\}| \leq \aleph_k$.

This is a corollary of (B) above.

7.2 Fragments of Zermelo set theory obtained by weakening the power set axiom

We define a sequence of set theories by recursion:

Z_0 is Zermelo set theory shorn of the power set axiom;

Z_{n+1} is Z_n + "If x is the same size as a von Neumann ordinal then $\mathcal{P}^{n+1}(x)$ exists"

(C) For $k = 0, 1, 2, 3 \ldots$ $\exists \omega_k$ is the axiom "the von Neumann ordinal ω_k exists".

Theorem 55 *For each $k = 0, 1 \ldots$ and each $n = 1, 2 \ldots$ we can prove $NF \vdash \langle M_{\omega_{k+1}}, \mathcal{E} \rangle \models Z_n + \exists \omega_0 + \exists \omega_1 + \ldots + \exists \omega_k$*

Proof:

Let us write 'M' instead of 'M_{ω_k}'. We know already that $\langle M, \mathcal{E} \rangle \models$Zermelo + TCl \ power set. (This was lemma 46 from chapter 4 and remark 49.)

Lemma (B) (page 51) implies the existence of this model M and of the existence within it of the von Neumann ordinals $\omega \ldots \omega_k$.

Now if $\mathfrak{a}$ is an element of M that M believes to be equipollent to a von Neumann ordinal then (by lemmas (B) and (C) page 51) $|\{\mathfrak{b} : \mathfrak{b}\ \mathcal{E}\ \mathfrak{a}\}| \leq \aleph_{k+1}$.

Since $\aleph_{k+1}$ is cantorian it follows that $|\{x \in dom(R) : xR(\mathbb{1}_R\})| \leq \aleph_{k+1}$, where $R \in \Omega$ and $\mathfrak{a} = \mathcal{T}(R)$. From this it follows that if $A = \{x \in dom(R) : xR(\mathbb{1}_R)\}$ there is an injection $f : A \to NO$ such that, for all $x \in A$, $f(x)$ is Tz for some z. Up to an isomorphism we may suppose that the elements of $dom(R)$ are of the form $\langle y, 0 \rangle$ and that, in particular, those elements $\mathfrak{b}$ of $dom(R)$ such that $\mathfrak{b}\ R\,(\mathbb{1}_R)$ are of the form $\langle Tz, 0 \rangle$ where z is an ordinal.

Next—since A is $\{\mathfrak{b} \in dom(R) : \mathfrak{b}\ R\,(\mathbb{1}_R\}$—one can associate to each $B \subseteq A$ the set $B' = \{z : \langle Tz, 0 \rangle \in B\}$. Manifestly $B' \in \mathcal{P}(NO)$. Also B' depends on B so there is a formula ϕ such that $B' = \phi(B)$, and moreover this formula is stratified with 'B' one type higher than 'B''. So let us define

$$S := \ R \cup \{\langle \mathfrak{b}, \langle \phi(B), 1 \rangle \rangle : \mathfrak{b} \in B \wedge B \subseteq A\} \cup \{\langle \langle \phi(B), 1 \rangle, \langle V, 2 \rangle \rangle : B \subseteq A\}$$

S is defined by a stratified set abstract. If $\mathfrak{t} = \mathcal{T}(S)$ we verify easily that $\mathfrak{t}$ is the power set of $\mathfrak{a}$ in the sense of $\langle M, \mathcal{E} \rangle$.

We have just shown that $\langle M, \mathcal{E} \rangle$ is a model of Z_1, and the proof that it is a model of Z_n for other [concrete] n is analogous. It suffices to bear in mind that whenever $\mathfrak{a} \in M$ and $\mathfrak{a} = \mathcal{T}(R)$ the assumption that $\mathfrak{a}$ is the same size as a von Neumann ordinal (in the sense of M) implies that if $A = \{x \in dom(R) : x\ R\,(\mathbb{1}_R)\}$ then A is the same size as a set of ordinals all of the form $T^{n+1}z$.

Corollary 56 *Con(NF)* $\to$ *Con*$(\bigcup_{n \in \mathbb{N}} Z_n + TCl + \exists \omega_0 + \ldots \exists \omega_n)$

Proof: Compactness. ∎

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