Université Libre de Bruxelles
Faculté des Sciences

Thèse de Doctorat – Mathématiques
Specialité : Logique Mathématique

MORE STUDIES ON THE AXIOM OF COMPREHENSION

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Soutenue le 4 juin 2004

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PREFACE

« One has often the impression that mathematicians talk about set theory as something unique. They appear to mean the Zermelo-Fraenkel theory. Of course the assumed uniqueness is illusory.»

Albert Thoralf Skolem (23 May 1887 – 23 March 1963),
in *'Studies on the Axiom of Comprehension'*: [38], §3, p. 167.

It is certainly the relativity inherent in any axiomatic theory - which he underscored and to which logicians are now accustomed - that forced Skolem to write in his last submitted paper these few words about the most accepted set theory, of which incidentally he is one of the founders. But for a mathematician and his unfailing curiosity, this relativeness is a gift. So it was presumably just to satisfy his curiosity that Skolem started investigating alternative semantics for naive set theory at the end of his life. The title of my thesis is a respectful allusion to his last pioneer work.

The main theme pervading this thesis is 'positive set theory', which, as we shall see, originates precisely in that work by Skolem. I shall show that this solution route to the set-theoretical paradoxes is sensitive to the use of an abstractor in the language; and the price to be paid for this is the loss of equality in formulas defining sets. Thereupon I shall also examine to my cost another solution route in which equality is blamed for the paradoxes. Both of these ways out hang together in sharing the existence of topological models, which is another permeating theme in this thesis.

The plan is briefly as follows. The first chapter is an introduction to Frege's original formulation of the logical notion of set, which does provide a good starting point as we can trace the use of an abstractor to Frege. For the most part, Chapter 2 may be regarded as an index of notations, though it contains some insights on positive formulas and topological models. In Chapter 3, I make a brief excursion into non-classical logic, relating the emergence of the distinction comprehension/abstraction in positive set theory and the existence of topological models. On my way, I revisit Skala's topological set theory in Chapter 4 on both the axiomatic and semantic sides. I show in Chapter 5 that the models of positive comprehension given by Skolem in [38] are in fact the natural topological solutions to the consistency problem for positive abstraction with extensionality. Finally, a term model solution to that problem is discussed in Chapter 6. It is not a good place here to give a more intelligible description of my investigations; anyway, I have devoted Chapter 3 to that.

It was initially planed to include a second part to this thesis, in which the paracomplete and the paraconsistent versions would have been discussed in details. But in view of the extent of the task, I have had to content myself with supplying the reader with a few bibliographical references, including my first publications on the subject.

One of my principal goals in writing this thesis was to show that it is really possible to have a comprehensive view of various alternative proposals. I have not been looking for a new one. I have just tried too to satisfy my curiosity, and as often in mathematics, this has proved to be enriching.

I would like to thank all the participants of our seminars, and especially those few fanatics of non-standard set theories. They will recognise themselves. Of course, I owe a special dept of gratitude to one of them in particular, my adviser, Roland Hinnion, for having let his curiosity rub off on me. May this thesis contaminate others.

CONTENTS

Chapter 1

FREGE'S THEORY OF CONCEPTS AND EXTENSIONS

Set theory was created by Georg Cantor, so we start with the 'definition' of the *naive* concept of set, as given in the final presentation of his lifework:

> « *A set is a collection into a whole of definite distinct objects of our intuition or of our thought. The objects are called the elements (members) of the set.*» [Translated from German.]

By 'into a whole' is meant the consideration of a set as an *entity*, an *abstract object*, which in turn can be collected to define other sets, etc. This *abstraction* step marks the birth of set theory as a mathematical discipline.

The *logical* formulation of the naive notion of set, however, was first explicitly presented at the end of 19th century by one of the founders of modern symbolic logic, Gottlob Frege, in his attempt to derive number theory from logic. As widely known, the resulting formal system was proved to be inconsistent by Russell in 1902.

In this introductory philosophically-oriented chapter, we briefly review some basic features of Frege's theory in order to frame and motivate our investigations.

1.1 The abstraction process

First of all, Frege's original predicate calculus is *second-order*. To simplify matters, let us say here that there are two types of variables ranging over mutually exclusive domains of discourse, one for *objects* $(u, v, \ldots)$, another for *concepts* $(P, Q, \ldots)$, where a *concept* P is defined to be any unary predicate $P(x)$ whose argument x ranges over *objects*.

Frege's system is characterized by a *type-lowering* correlation: with each concept P is associated an abstract object, the *extension* of the concept, which is now familiarly denoted by $\{x \mid P\}$, and is meant to be the collection of all objects x that *fall under* the concept P. This correspondence between concepts and objects is governed by the following principle, known as

Basic Law V:

$$\forall P \forall Q\ (\ \{x \mid P\} = \{x \mid Q\} \ \longleftrightarrow\ \forall u(P(u) \equiv Q(u))\ ).$$

The *equality* symbol = on the left-hand side is the identity between objects, which Frege takes as primitive in his language. The right side is the *material equivalence* of concepts, where $\equiv$ is an abbreviation for 'having the same truth value'. This may be the material biconditional $\leftrightarrow$, but in Frege's notation this is again =. The reason is that in his system *predication* is understood as *functional application*. To do that Frege naturally selects two special objects he calls *truth-values* and he just defines a concept to be any function that maps objects to truth values. Accordingly, it may be more perspicuous to denote the extension of a concept P by $\lambda x\, P$, using notation from the λ-calculus, and think of it as the *graph* of the function defining it, which Frege calls *course-of-values*.[1] Note that Frege insists on a rigid distinction between functions and objects: a concept is *not* an object; only its extension is.

Whether a concept be looked at as a predicate or as a (truth-)function, we shall call this objectification of concepts *abstraction*. It has rarely been emphasized that Frege internalizes this process in the language by explicitly making use of an *abstraction operator* to *name* the extension of a concept. Whether denoted by $\{\cdot \mid -\}$ or by $\lambda \cdot -$, the use of such an *abstractor* in the language of set theory is one source of investigation in this thesis.

1.2 Sets and membership

Those objects that are extensions of concepts are called *sets*. Frege then defines what it is for an object to be a *member* of a set: u is a member of v, denoted by $u \in v$, if and only if u falls under a concept of which v is the extension, i.e., $\exists P(v = \{x \mid P\} \wedge P(u))$.[2] Note incidentally that both second-order and the use of the abstractor are required for that definition, or for the one of the concept 'being a set', that is, $Set(v) :\equiv \exists P(v = \{x \mid P\})$.

Given the definition of membership, an immediate consequence of Basic Law V is the

Law of Extensions:

$$\forall P\ \forall u(u \in \{x \mid P\} \equiv P(u))$$

[1] By the way, Frege's original notation for the extension of a concept is something like $\grave{\varepsilon} P(\varepsilon)$, which is indeed the embryonic version of the present λ-notation.

[2] The epsilon notation '$\in$' is due to Peano. Frege used something similar to '$\cap$' to designate the membership relation.

from which by Existential Introduction follows the well-known

Principle of Naive Comprehension:

$$\forall P\ \exists v \forall u(u \in v \equiv P(u)).$$

According to the Law of Extensions, '$\in$' may just be regarded as an allegory for *predication*, this latter being now a proper object of the language. But using the λ-notation, membership should rather be understood as *application* '$\cdot$', so that the Law of Extensions would correspond to the

Principle of λ-Conversion:

$$\forall P\ \forall u(\lambda x\, P \cdot u = P(u)).$$

Another significant rule derivable from Basic Law V is the

Principle of Extensionality:

$$\forall v \forall w(Set(v) \wedge Set(w) \longrightarrow (\forall u(u \in v \equiv u \in w) \rightarrow v = w)).$$

Sets, thought of as collections, are thus completely determined by their members. By combining the Law of Extensions and the Principle of Extensionality, it is shown that any set v is at least the extension of the concept $P(x) :\equiv x \in v$, i.e.: $\forall v(Set(v) \rightarrow\ v = \{x \mid x \in v\})$. Note that there is no presumption that all objects are sets. As our aim is merely to study pure and abstract set-theoretic systems, we shall however assume this from now on, that is to say, $\forall v\, Set(v)$.

1.3 First-order versions

Second-order logic and the use of an abstractor are by no means necessary to render an account of naive set theory. First-order versions of Frege's calculus are obtained by taking $\in$ as primitive notion in the language, retaining the Principle of Extensionality, and restricting either the Law of Extensions or the Principle of Naive Comprehension to concepts definable by first-order formulas (possibly with parameters).

In choosing the Law of Extensions the language is still assumed to be equipped with an abstractor, which yields what we call the

Abstraction Scheme:

For each formula $\varphi(x)$ of the language *with* abstractor,

$$\forall u(u \in \{x \mid \varphi\} \equiv \varphi(u)).$$

By the choice of the Principle of Naive Comprehension, it is understood that the language is no longer equipped with an abstractor, which gives the

Comprehension Scheme:
For any formula $\varphi(x)$ of the language *without* abstractor,
$\exists v \forall u(u \in v \equiv \varphi(u))$.

First-order comprehension with extensionality is often presented as the *ideal* formalization of set theory. However that may be, it is inconsistent. One of our goals is to clearly distinguish abstraction from comprehension in a specific *consistent* context.

1.4 Russell's paradox

Set Theory originated in Cantor's result showing that some infinities are definitely bigger than others. Paradoxically enough, it is precisely this rather positive result that resulted in the inconsistency of Frege's system, and so in the incoherence of naive set theory.

Cantor proved, by its famous *diagonal argument*, that the domain of all single-valued functions that map any given domain of discourse U to the two-elements set $\{0, 1\}$ cannot be put into one-to-one correspondence to U.[3] But this clearly contradicted what the left-to-right direction of Basic Law V was asserting, at least in its original *second-order* formulation.

Inspired by Cantor's diagonal argument, Russell finally presented an elementary proof of the incoherence of naive set theory by pointing out that the mere existence of $\{x \mid x \notin x\}$ is simply and irrevocably devastating. Still more dramatically, thinking of membership as predication, as hinted above, one could reformulate the theory of concepts and extensions without even explicitly referring to the *mathematical* concept of set as collection. That Russell's paradox could be so formulated in terms of most basic *logical* concepts came as a shock.

1.5 Solution routes

If one believes in the soundness of logic as used in mathematics throughout the ages, then one must admit that some collections are not '*objectifiable*'. The decision as to which concepts to disqualify or disregard is as difficult as it is counter-intuitive. This is attested by the diversity of diagnoses and systems advocated.

[3] It is to be noticed that Cantor did not use power sets. Just like Frege, he was making use of (and expanding) the notion of *function* to develop his theory.

Roughly, the various proposals may be divided into two categories according to whether $\{x \mid x \in x\}$ is accepted as a set or not. This distinction is, of course, more emblematic than well-established.

The second category encompasses the so-called *type-theoretic* approaches, those involving *syntactical* criteria to select admissible concepts by prohibiting circularity in definitions. In this thesis we will rather be concerned with *type-free* approaches, and mainly with ones that belong to the first category, though we shall be led to make some incursions into the second as well.

Within those systems admitting $\{x \mid x \in x\}$ as a set there is no alternative but to tamper with the use of $\neg$ or with the definition of $\equiv$. It is the former alternative that is explored in details throughout this thesis.

Chapter 2

LANGUAGE AND SET-THEORETIC STRUCTURES

We cannot in good conscience begin without mentioning that throughout this thesis we shall be assuming ZF as underlying theory. ZF^- stands for ZF without the Foundation Axiom, and ZFA for ZF^-+ the Anti-Foundation Axiom (as in [2] & [7]). Note that we always assume the Axiom of Choice. Should we need it, the use of the Continuum Hypothesis or of any large cardinal assumption will be stipulated.

In this chapter we set our notations and conventions concerning the language of set theory in general, the ground theory in particular and the way set-theoretic structures are represented in it. We do hope that this irksome task will make the reading of the next chapters easier. The last section is by far more attractive as it provides explanations and motivations on what we are going to do in these.

2.1 The language of set theory

We start with a few notational conventions regarding the use of variables within the first-order predicate calculus.

The letters $x, y, z, \dots / p, q, r, \dots$ - also with subscripts, superscripts, etc. - stand for variables of the *object-language* as well as for meta-variables ranging over these (under the assumption that different letters stand for different variables), and in particular we will reserve the letters p, q, r for parameters. The Greek letters φ, ψ, χ and τ, σ, ρ are meta-variables for formulas and terms respectively. We use the overline notation for finite lists of variables of a given sort, e.g. $\bar{x}$ for $x_1, \dots, x_n$.

By introducing a formula φ or a term τ as $\varphi(\bar{x})$ or $\tau(\bar{x})$, we want to underline the free occurrences (possibly none) of each of the variables $\bar{x}$ in φ or τ. Note that φ or τ may have free variables other than $\bar{x}$; those additional variables are called *parameters*. When we want to specify parameters too, we write $\varphi(\bar{x}, \bar{p})$ or $\tau(\bar{x}, \bar{p})$ - in particular $\varphi(\bar{p})$ or $\tau(\bar{p})$ if $\bar{x}$ is empty; all the free variables of φ or τ are then supposed to be among $\bar{x}, \bar{p}$.

Later in the same context, given $\varphi(\bar{x})$ or $\tau(\bar{x})$ and a list of terms $\bar{\sigma}$ of the same length as $\bar{x}$, we may write $\varphi(\bar{\sigma})$ or $\tau(\bar{\sigma})$ to designate the result of simultaneously replacing each free occurrence of x_i in φ or τ by σ_i for $i = 1, \ldots, n$. And to avoid collisions of variables in substitutions, we may assume that bound variables in formulas and terms have been replaced by a ghost letter, at least formally, so that two formulas or two terms that only differ in the name of their bound variables are regarded as identical.[1]

Finally, to deal with valuations in an easy way, we may also assume that, given an interpretation, the object-language has temporarily been extended by constants naming the elements of the domain; these constants may be the elements of the domain themselves.

We now turn to a thorough examination of the language of set theory.

The basic language

What we denote by $\mathscr{L}$ is the language in first-order predicate calculus *with equality* whose the only non-logical primitive symbol is $\in$. *Adding '$*$' as subscript means that $=$ is excluded from formulas.* It is useful to take all the usual logical connectives $\neg, \wedge, \vee, \rightarrow, \equiv, \forall, \exists$ as primitive, for in the next chapter we will be considering non-classical interpretations of these as well. Unless otherwise explicitly stated, the underlying logic is supposed to be classical and the interpretation of the connectives standard. It is then suitable to incorporate in $\mathscr{L}$ two propositional constants $\bot$ and $\top$ denoting any fixed identically false statement and its negation.

Abstraction terms

In developing any *extensional* set theory it is common practice to virtually extend the language $\mathscr{L}$ by introducing *abstraction terms* '$\{x \mid \varphi(x)\}$', either as *defined* terms (possibly with parameters) or as a *façon de parler*. We emphasize that in both cases these are not proper symbols of $\mathscr{L}$.[2] These are introduced as abbreviations to make certain expressions more readable or easier to handle. They can be eliminated in accordance with their intended

[1] For instance one might use Bourbaki's square in formally defining the language, and code formulas and terms as finite sequences in the metatheory.

[2] In the second case they need not even be denoting objects of the theory. When a set abstract is proved to be denoting, we may omit the quotation marks ' '.

meaning by applying successively the following rules:

$$\begin{array}{lll} R1. & `\{x \mid \varphi(x)\}\text{'} \in \tau & \Leftrightarrow \quad \exists z(z = `\{x \mid \varphi(x)\}\text{'} \wedge z \in \tau) \\ R2. & \tau = `\{x \mid \varphi(x)\}\text{'} & \Leftrightarrow \quad \forall z(z \in \tau \leftrightarrow \varphi(z)) \\ R3. & z \in `\{x \mid \varphi(x)\}\text{'} & \Leftrightarrow \quad \varphi(z) \end{array}$$

where in $R1$ & $R2$ τ stands for any variable or for an abstraction term.[3]

Comprehension

Using these abbreviations, an instance of the comprehension scheme may be more suggestively restated like this:

$$Comp[\varphi(x)] \ : \quad \exists y(y = `\{x \mid \varphi(x)\}\text{'}).$$

Here it is usually understood that the variable y does not occur free in φ.[4] The removal of this familiar restriction, however, is not meaningless. It gives rise to what we call *reflexive* comprehension, a typical instance of which is:

$$Comp_{\circlearrowleft}[\varphi(x,y)] \ : \quad \exists y(y = `\{x \mid \varphi(x,y)\}\text{'}).$$

Obviously, whenever y is not free in φ, we recover classical comprehension.

Assuming extensionality, the y provided by $Comp[\varphi(x)]$ is unique, and '$\{x \mid \varphi(x)\}$' can thus be defined as a name for it; but there is no guarantee that the y given by $Comp_{\circlearrowleft}[\varphi(x,y)]$ is. There might be many such y's. For instance, if we take φ to be $x \in y$, then *any* set y is solution to $Comp_{\circlearrowleft}[\varphi(x,y)]$.[5]

Accordingly, we may consider the use of a syntactical device to name, in an *uniform* way, one solution to each instance of the comprehension scheme under consideration.

The extended language

What we designate by $\mathscr{L}_\tau$ is the actual extension of $\mathscr{L}$ obtained by adjoining an *abstractor* $\{\cdot \mid -\}$ so as to allow the formation of set abstracts as *primitive*

[3] Of course, the new variable z introduced in $R1$ & $R2$ is supposed not to be occurring free in τ or φ. Mention of such obvious precautions will be omitted henceforth. Note also that according to $R1$, the occurrence of an abstraction term on the left side of $\in$ compels it to be denoting.

[4] Cf. previous footnote.

[5] For a more elaborated example, we may instance the result according to which if ZF^- is consistent, so is ZF^- + the existence of infinitely many Quine atoms, where a Quine atom is just defined to be a set y such that $y = `\{x \mid x = y\}\text{'}$. On the other hand, there is one and only Quine atom in ZFA, and none in ZF.

terms of the language. The rule governing the use of this term-forming operator is the following:

> for any distinct variables x & y and any $\mathscr{L}_\tau$-formula $\varphi(x,y)$, $\{x \mid_y \varphi\}$ is an $\mathscr{L}_\tau$-term in which x & y are bound and all the remaining variables occurring free in φ are taken to be free.

Note that according to our convention regarding bound variables, $\{x \mid_y \varphi\}$ and $\{x' \mid_{y'} \psi\}$, where ψ is $\varphi(x', y')$, are identical. We let $\{x \mid \varphi\}$ stand for any $\{x \mid_y \varphi\}$ where y does not occur free in φ.

Abstraction

Within this extended language, we shall speak of *abstraction* instead of comprehension to stress that set abstracts may already appear as terms in the formula φ involved in the corresponding instance, which is formulated by

$$\mathit{Abst}[\varphi(x)] \ : \quad \{x \mid \varphi\} = \text{`}\{x \mid \varphi(x)\}\text{'}$$

or, for the reflexive version, by

$$\mathit{Abst}_{\circlearrowleft}[\varphi(x,y)] \ : \quad \{x \mid_y \varphi\} = \text{`}\{x \mid \varphi(x, \{x \mid_y \varphi\})\}\text{'}\,.$$

The use of abstraction terms here is subject to the same rules as above, according to which simple set abstracts $\{x \mid \varphi\}$ could thus be eliminated, but not reflexive ones, as $\{x \mid_y x \in y\}$ would show.

In any consistent situation, not every set abstract can be denoting so that the abstractor can only be *partially* defined. We are going to study an example of such a situation in which in particular we can draw a clear distinction between comprehension and abstraction.

In places we will make use of the λ-notation discussed in Chapter 1. The corresponding notations for $\{x \mid \varphi\}$ and $\{x \mid \varphi_y\}$ are $\lambda x\, \varphi$ and $\lambda_y x\, \varphi$. It is there also appropriate to replace membership '$\in$' by application '$\cdot$' in the language.

Some useful abbreviations

We end Section 2.1 by introducing names for certain abstraction terms. These are mainly used in Chapter 4. Let us agree that whenever $\mathscr{X}$ is a name for '$\{x \mid \varphi(x)\}$', $(\mathscr{X})$ will stand for (the universal closure of) $\mathit{Comp}[\varphi(x)]$.

Attached to propositional constants and atomic formulas are the following abstractions terms and their names:

$$\Lambda := \text{`}\{x \mid \bot\}\text{'} \qquad \mathrm{W} := \text{`}\{x \mid x \in x\}\text{'} \qquad \mathrm{V} := \text{`}\{x \mid \top\}\text{'}$$

$$\mathscr{A}(p) := \text{`}\{x \mid p = x\}\text{'} \qquad \mathscr{B}(p) := \text{`}\{x \mid p \in x\}\text{'}\,.$$

Note that $\mathscr{A}(p)$ is equally defined by '$\{x \mid x = p\}$' and that '$\{x \mid x \in p\}$' $= p$.[6]

Beside $\in$ and $=$ two non-primitive binary relations deserve a particular interest. These are defined as follows:

$$x \underset{\cdot}{\leqslant} y \Leftrightarrow \forall z(z \in x \to z \in y) \qquad x \overset{\cdot}{\eqslantless} y \Leftrightarrow \forall z(x \in z \to y \in z).$$

The relationship between these and the primitive symbols $\in$ and $=$ may be underlined by the following observation.

Fact 2.1.1. *$x \in y \Leftrightarrow \mathscr{A}(x) \underset{\cdot}{\leqslant} y$ and, assuming $(\mathscr{A})$, $x = y \Leftrightarrow x \overset{\cdot}{\eqslantless} y$.*

Whether $(\mathscr{A})$ is assumed or not, $\underset{\cdot}{\leqslant}$ has a major part in any set theory, as it is, of course, the *inclusion* relation, commonly denoted by $\subseteq$.[7]

However odd that may seem, we are going to examine situations in which $(\mathscr{A})$ fails. According to Fact 2.1.1, in such a situation $\overset{\cdot}{\eqslantless}$ may be thought of as a *reminiscence* of the equality; this will be underscored in Chapter 4.

We now introduce the following abstraction terms attached to $\underset{\cdot}{\leqslant}$ and $\overset{\cdot}{\eqslantless}$:

$$\mathscr{P}(p) := \text{`}\{x \mid x \underset{\cdot}{\leqslant} p\}\text{'} \qquad \mathscr{M}(p) := \text{`}\{x \mid p \overset{\cdot}{\eqslantless} x\}\text{'}\,.$$

$\mathscr{P}(\cdot)$ is the well-known *power-set* operator. The role of $\mathscr{M}(\cdot)$ is going to have to be explained (see Chapter 4). Neither '$\{x \mid p \underset{\cdot}{\leqslant} x\}$' nor '$\{x \mid x \overset{\cdot}{\eqslantless} p\}$' will be of interest to us in our investigations.

By the way, we also introduce the abstraction term corresponding to *complementation*, to which we reserve a particular treatment:

$$\mathscr{C}(p) := \text{`}\{x \mid x \notin p\}\text{'}\,.$$

At last, we point out two useful abbreviations, which are sometimes referred to as the *extensional* and the *intensional* equalities respectively:

$$x \underset{\cdot}{=} y \Leftrightarrow x \underset{\cdot}{\leqslant} y \land y \underset{\cdot}{\leqslant} x \qquad x \doteq y \Leftrightarrow x \overset{\cdot}{\eqslantless} y \land y \overset{\cdot}{\eqslantless} x.$$

In any extensional set theory, $=$ must coincide with $\underset{\cdot}{=}$, so that the equality might not be taken as primitive symbol. Note that, in the absence of $=$ in the language, extensionality should be formulated by $\forall x \forall y(x \underset{\cdot}{=} y \to x \doteq y)$, which guarantees that $\underset{\cdot}{=}$ has the substitutivity property for $\mathscr{L}_*$-formulas. It is worth noticing that the converse of this, namely $\forall x \forall y(x \doteq y \to x \underset{\cdot}{=} y)$, holds in any set theory satisfying $(\mathscr{B})$ and, still more obviously, in any one satisfying $(\mathscr{A})$, as then $\doteq$ must be $=$ (more comments on this in Chapter 4).

[6] For this latter extensionality is required.

[7] We will reserve this notation for the meta-theory.

2.2 About the ground theory

As we said at the beginning, we shall be working within ZF as meta-theory. We assume all the customary practices of that set theory in the background, as the relaxed use of the 'set of' operator $\{\,\}$, etc. The reader will notice that we use (as much as possible) a smaller $\in$ for the meta-theoretic membership.

We want to introduce here our particular notations as regards functions. We would remind the reader that set-theoretically a function f is defined by its *graph*, that is, the set of ordered pairs $(x, f(x))$. Thus functions are just particular sets of couples, namely *(binary) relations.* Some special notations will apply to relations, and then to functions.[8]

As usual, given a set R of ordered pairs, we write $x\ R\ y$ for $(x, y) \in R$. The *domain* and the *range* of R are defined by $\mathrm{dom}R := \{x \mid \exists y : x\ R\ y\}$ and $\mathrm{rng}R := \mathrm{dom}R^{-1}$, where $R^{-1} := \{(x, y) \mid y\ R\ x\}$; and given a set E, we denote the *image* of E under R by $R``E := \{y \mid \exists x \in E : x\ R\ y\}$.

When f is a function, the *value* of f at x is denoted by $f`x$, which is thus defined by $\{f`x\} = f``\{x\}$, so that *functional application* in the meta-theory is preferably written $f`x$ instead of $f(x)$. We naturally extend this notation to list of variables, say $\bar{x} = x_1, \ldots, x_n$, by setting $f`\bar{x} := f`x_1, \ldots, f`x_n$. This is not to be confused with $f`(\bar{x})$ or $f(\bar{x})$ in case f has many variables, i.e., when $\mathrm{dom}f$ is a set of tuples $(\bar{x}) = (x_1, \ldots, x_n)$. Note incidentally that we will tacitly use all the habitual labor-saving devices in handling tuples and cartesian products. Thus, though tuples are formally defined as ordered pairs in ZF, it may be more convenient in some circumstances to look at them as finite sequences, so that for instance we may identify the cartesian products $A \times (B \times C)$ and $(A \times B) \times C$, and then simply write $A \times B \times C$, etc.

For *functional abstraction* in the meta-theory, we agree that any function f may equally be designated by $x \mapsto f`x$. This enables us to easily define new functions from old. For instance, if we are given a two-variables function $(x, y) \mapsto f`(x, y)$ and $b \in \mathrm{rng}(\mathrm{dom}f)$, then $g : x \mapsto f`(x, b)$ will represent the function $g = \{(a, f`(a, b)) \mid a \in \mathrm{dom}(\mathrm{dom}f)\}$.

The notation $f : A \longrightarrow B$ means that f is a function with $\mathrm{dom}f = A$ and $\mathrm{rng}f \subseteq B$, and we denote the set of all functions $f : A \longrightarrow B$ by $(A \rightarrow B)$. The notation β^α is reserved for exponentiation of cardinals $\alpha = |A|, \beta = |B|$. We write $A \simeq B$ to indicate that two sets A and B have the same cardinal. If a particular kind of structure has been specified on these and we want to express that they are isomorphic in the corresponding category of structured sets, we write $A \cong B$. Thus $\simeq$ does correspond to $\cong$ in the category of pure sets, which we call SET.

[8] These notations may apply as well to relations and functions that are proper classes.

There is a correspondence in *SET* that is worth making explicit, namely the canonical bijection between the *power-object* $\mathscr{P}(A)$ and the *exponential* $(A \to 2)$, where $2 = \{0, 1\}$. It will be designated by $\imath_A : \mathscr{P}(A) \longrightarrow (A \to 2)$ and its inverse by $\jmath_A : (A \to 2) \longrightarrow \mathscr{P}(A)$, i.e.

$$\text{for any } E \in \mathscr{P}(A), \quad \imath_A`E : x \longmapsto \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$$

and

$$\text{for any } f \in (A \to 2), \quad \jmath_A`f := \{x \in A \mid f`x = 1\}\,.$$

Although they are isomorphic in *SET*, there is a difference from the categorical point of view between $\mathscr{P}(A)$ and $(A \to 2)$, in that the endofunctor $\mathscr{P}(\cdot)$ is *covariant* whereas $(\cdot \to 2)$ is *contravariant*: when $\mathcal{F}(\cdot)$ is $\mathscr{P}(\cdot)$ the action of $\mathcal{F}(\cdot)$ on $f : A \longrightarrow B$ is defined by $f^{\mathcal{F}} : \mathscr{P}(A) \longrightarrow \mathscr{P}(B) : E \longmapsto f``E$; when $\mathcal{F}(\cdot)$ is $(\cdot \to 2)$ it is given by $f^{\mathcal{F}} : (B \to 2) \longrightarrow (A \to 2) : h \longmapsto h \circ f$. Thus $(g \circ f)^{\mathcal{F}} = g^{\mathcal{F}} \circ f^{\mathcal{F}}$ in the first case, whereas $(g \circ f)^{\mathcal{F}} = f^{\mathcal{F}} \circ g^{\mathcal{F}}$ in the second. (As usual, we use '$\circ$' for composition of functions or relations.)

2.3 Set-theoretic structures

Here we present different ways of modelling a set-theoretic structure. As that will be illustrated throughout this thesis, one view may prove more suitable than another depending on the context. We also have a look at corresponding homomorphisms and the formulas they preserve.

Structures

As an $\mathscr{L}$-structure, a set-theoretic universe $\mathcal{U}$ is regarded as

$$\langle U; \in_{\mathcal{U}} \rangle \quad \text{where} \quad \in_{\mathcal{U}} \subseteq U \times U \qquad\qquad (U \neq \emptyset).$$

But if a set is to be conceived of as the collection of its members, the following equivalent presentation is arguably more natural:

$$\langle U; [\cdot]_{\mathcal{U}} \rangle \quad \text{where} \quad [\cdot]_{\mathcal{U}} : \begin{array}{l} U \longrightarrow \mathscr{P}(U) \\ v \longmapsto \{u \in U \mid u \in_{\mathcal{U}} v\}. \end{array}$$

We call $[y]_{\mathcal{U}}$ the *extension* of y in $\mathcal{U}$.[9]

[9] It is worth stressing the difference between Frege's definition of the *extension of a concept*, which is the corresponding set as object, and the *extension of a set* in a given structure as defined here.

We now display the notations for the valued counterparts of these settings:

$$\langle U; \epsilon_{\mathcal{U}} \rangle \quad \text{where} \quad \epsilon_{\mathcal{U}} : \quad U \times U \longrightarrow 2$$

and

$$\langle U; [\![\cdot]\!]_{\mathcal{U}} \rangle \quad \text{where} \quad [\![\cdot]\!]_{\mathcal{U}} : \begin{array}{l} U \longrightarrow (U \to 2) \\ v \longmapsto (u \mapsto \epsilon_{\mathcal{U}}`(u,v)) . \end{array}$$

Explicitly, $\epsilon_{\mathcal{U}} = \imath_{U \times U}`\in_{\mathcal{U}}$ and $[\![v]\!]_{\mathcal{U}} = \imath_U`[v]_{\mathcal{U}}$ for every $v \in U$.

Unless otherwise stated, the interpretation of the equality relation in a structure is taken to be the *identity*, i.e.

$$\Delta_U := \{(u,v) \in U \times U \mid u = v\} \;/\; \delta_U := \imath_{U \times U}`\Delta_{\mathcal{U}} : (u,v) \longmapsto \begin{cases} 1 & \text{if } u = v \\ 0 & \text{otherwise.} \end{cases}$$

This *canonical* interpretation may not always be appropriate (as illustrated in Chapter 6). Nevertheless, it is well known that any structure with an *acceptable* interpretation of $=$ can be *contracted* to a *normal* one, that is, one in which $=$ is the identity.[10] As far as extensionality is concerned, the extension function $[\cdot]_{\mathcal{U}}/[\![\cdot]\!]_{\mathcal{U}}$ is injective when $\mathcal{U}$ is normal.

Language in structures

As previously mentioned, given a structure $\mathcal{U}$, we conveniently extend the language by the elements of U seen as constants; this extended language is designated by $\mathscr{L}(U)$ or $\mathscr{L}_\tau(U)$. For this latter we will use the λ-notation whenever structures are being looked at in the valued setting(s).

We denote the *truth-value* of a formula $\varphi(\bar{p})$ interpreted in $\mathcal{U}$, for the assignation $\bar{p} := \bar{u}$ in U, by $|\varphi(\bar{u})|_{\mathcal{U}}$. The interpretation of a term $\tau(\bar{p})$ within a structure $\mathcal{U}$, for $\bar{p} := \bar{u}$ in U, is designated by $\tau^{\mathcal{U}}(\bar{u})$.

What is needed to turn an $\mathscr{L}$-structure into an $\mathscr{L}_\tau$-one is the interpretation of the abstractor wherever it is defined, and this actually depends on the fragment of $\mathscr{L}_\tau$ under consideration.

Besides being subject to satisfying the corresponding instances of abstraction, the interpretation of $\{\cdot \mid -\}$ in a structure $\mathcal{U}$ has to fulfil a natural substitutivity property. Namely, for any $\mathscr{L}_\tau$-formula $\varphi(x, \bar{p})$ and list of $\mathscr{L}_\tau$-terms $\bar{\tau}(\bar{q})$ of the same length as $\bar{p}$, it is required that, whenever all the terms involved are defined, the following equality hold:

$$\mathcal{U} \models \{x \mid \varphi\}^{\mathcal{U}}(\bar{\tau}^{\mathcal{U}}(\bar{u})) = \{x \mid \psi\}^{\mathcal{U}}(\bar{u})$$

[10] By an ‘acceptable’ interpretation of $=$ we mean one that ensures the substitutivity property for formulas and terms of the language under consideration.

for any $\bar{u}$ in U of the same length as $\bar{q}$, and where ψ is the formula $\varphi(x, \bar{\tau}(\bar{q}))$ (and likewise for reflexive set abstracts).

Basically, the interpretation of the abstractor in a (normal) structure $\mathcal{U}$ may be conceived of as the 'inverse' of the extension function $[\cdot]_{\mathcal{U}} / [\![\cdot]\!]_{\mathcal{U}}$ (we are also assuming extensionality here). For this to be apparent, let us define the *pre-extension* of a formula $\varphi(\bar{x}, \bar{p})$ in $\mathcal{U}$, for $\bar{p} := \bar{v}$ in U, by

$$\langle \bar{x} \mid \varphi(\bar{x}, \bar{v}) \rangle_{\mathcal{U}} := \{(\bar{u}) \in U^n \mid \mathcal{U} \models \varphi(\bar{u}, \bar{v})\}$$

or, for the valued version, by

$$\langle\!\langle \bar{x} \mid \varphi(\bar{x}, \bar{v}) \rangle\!\rangle_{\mathcal{U}} := \imath_{U^n}`\langle \bar{x} \mid \varphi(\bar{x}, \bar{v}) \rangle_{\mathcal{U}} : (\bar{u}) \mapsto |\varphi(\bar{u}, \bar{v})|_{\mathcal{U}} .$$

Thus, for instance, we have $\langle x, y \mid x \in y \rangle_{\mathcal{U}} = \in_{\mathcal{U}}$ / $\langle\!\langle x, y \mid y \cdot x \rangle\!\rangle_{\mathcal{U}} = \epsilon_{\mathcal{U}}$, and for each $v \in U$, $\langle x \mid x \in v \rangle_{\mathcal{U}} = [v]_{\mathcal{U}}$ / $\langle\!\langle x \mid v \cdot x \rangle\!\rangle_{\mathcal{U}} = [\![v]\!]_{\mathcal{U}}$.

Now it is clear that

$$\{x \mid \varphi\}^{\mathcal{U}}(\bar{v}) = [\langle x \mid \varphi(x, \bar{v}) \rangle_{\mathcal{U}}]^{-1}_{\mathcal{U}} \quad / \quad (\lambda x\, \varphi)^{\mathcal{U}}(\bar{v}) = [\![\langle\!\langle x \mid \varphi(x, \bar{v}) \rangle\!\rangle_{\mathcal{U}}]\!]^{-1}_{\mathcal{U}}$$

and in particular $\{x \mid x \in p\}^{\mathcal{U}}(v) = v$ / $(\lambda x\, p \cdot x)^{\mathcal{U}}(v) = v$, as required.

Homomorphisms and the formulas they preserve

A *homomorphism* of $\mathscr{L}$-structures is defined to be a function $f : U \longrightarrow V$ such that for all $u, v \in U$, $u \in_{\mathcal{U}} v \Rightarrow f`u \in_{\mathcal{V}} f`v$, i.e. $\epsilon_{\mathcal{U}}`(u, v) \leqslant \epsilon_{\mathcal{V}}`(f`u, f`v)$. In this definition it is not required of a homomorphism that it preserve the notion of extension; all we have is $f``[v]_{\mathcal{U}} \subseteq [f`v]_{\mathcal{V}}$. So we shall say that $f$ is a $[\cdot]$*-homomorphism* if $f``[v]_{\mathcal{U}} = [f`v]_{\mathcal{V}}$, for each $v \in U$. To clearly distinguish between these two notions of homomorphism, let us have a look at the formulas they preserve. We say that a function $f : U \longrightarrow V$ preserves $\varphi(\bar{p})$ if for any $\bar{u}$ in U, $\mathcal{U} \models \varphi(\bar{u}) \Rightarrow \mathcal{V} \models \varphi(f`\bar{u})$, i.e. $|\varphi(\bar{u})|_{\mathcal{U}} \leqslant |\varphi(f`\bar{u})|_{\mathcal{V}}$.

On the one hand, it is well known that a homomorphism of $\mathscr{L}$-structures preserves *positive existential* $\mathscr{L}$-formulas, i.e. those built up from $\bot, \top$ and atomic formulas by using $\vee, \wedge, \exists$ only; in case the homomorphism is surjective, $\forall$ is allowed to occur, and this defines the set of *positive* $\mathscr{L}$-formulas, which is denoted by $\mathscr{L}^+$. Likewise, we recursively define the set $\mathscr{L}_\tau{}^+$ of *positive* $\mathscr{L}_\tau$-formulas by restricting the abstractor to formulas in $\mathscr{L}_\tau{}^+$. Note that if we require a surjective homomorphism f to preserve $\mathscr{L}_\tau{}^+$-terms, i.e. for every $\varphi(x, \bar{p})$ in $\mathscr{L}_\tau{}^+$ and any $\bar{u}$ in U, $f`\{x \mid \varphi\}^{\mathcal{U}}(\bar{u}) = \{x \mid \varphi\}^{\mathcal{V}}(f`\bar{u})$, then we get the same preservation result for $\mathscr{L}_\tau{}^+$-formulas.

On the other hand, it is proved that a surjective $[\cdot]$-homomorphism does preserve a larger class of $\mathscr{L}$-formulas, the set of so-called *bounded positive* formulas, denoted by $\mathscr{L}^{[+]}$, in which bounded quantifications of the form '$\forall x \in y$' are permitted too.[11] As a distinguished member of $\mathscr{L}^{[+]}$, we quote the formula $p \leqslant q$, and so $p \doteq q$. Given a $[\cdot]$-homomorphism $f : U \longrightarrow V$, let us show that if $\varphi(x, \bar{p})$ is preserved under f, so is $\forall x(x \in p_k \rightarrow \varphi(x, \bar{p}))$, where p_k is in $\bar{p}$.

Proof. Assume that $\varphi(x, \bar{p})$ is preserved and that $\mathcal{U} \models \forall x(x \in u_k \rightarrow \varphi(x, \bar{u}))$, for $\bar{u}$ given in U (u_k in $\bar{u}$). Let $v \in V$ such that $v \in_{\mathcal{V}} f`u_k$, that is, $v \in [f`u_k]_{\mathcal{V}}$. As f is a $[\cdot]$-homomorphism, we can find $u' \in [u_k]_{\mathcal{U}}$ such that $f`u' = v$. Now, from $u' \in_{\mathcal{U}} u_k$, we get $\mathcal{U} \models \varphi(u', \bar{u})$, and as $\varphi(x, \bar{p})$ is preserved, we have $\mathcal{V} \models \varphi(v, f`\bar{u})$. Whence $\mathcal{V} \models \forall x(x \in f`u_k \rightarrow \varphi(x, f`\bar{u}))$. ⊣

We have thus introduced the classes $\mathscr{L}^+/\mathscr{L}_\tau{}^+/\mathscr{L}^{[+]}$ in a natural way by looking at formulas that are preserved under *projections*. In the next chapter we will review what is known on comprehension/abstraction restricted to these classes of formulas.

To give a complete account, we now conclude this section by a couple of remarks related to homomorphisms again.

Remark 2.3.1. Assume we are given a set-theoretic structure $\mathcal{U}$ together with an equivalence relation R on U. There is a natural way to try to define a notion of extension on the quotient set $V := U/_R$. Namely, we define $[f`v]_{\mathcal{V}}$ to be $f``[v]_{\mathcal{U}}$, where $f : U \longrightarrow V : u \longmapsto R``\{u\}$ is the corresponding projection map. In order for this to work, it is necessary that for all $v, v' \in U$, $v\ R\ v' \Rightarrow R``[v]_{\mathcal{U}} = R``[v']_{\mathcal{U}}$, and as R is an equivalence relation, this amounts to $v\ R\ v' \Rightarrow [v]_{\mathcal{U}} \subseteq R``[v']_{\mathcal{U}}$. In theoretical computer science an equivalence relation R on U satisfying this condition is called a *(bi)simulation*. It is now apparent that bisimulations are exactly *kernels* of surjective $[\cdot]$-homomorphisms (where the kernel of a map f is as usual the equivalence relation R on $\mathrm{dom} f$ defined by $u\ R\ v$ if and only if $f`u = f`v$).

Remark 2.3.2. The reader familiar with category theory will have noticed that our notions of $[\cdot]/[\![\cdot]\!]$-structures are particular cases of *coalgebras*: given a category CAT and an endofunctor $\mathcal{F}$ acting on CAT, a $\mathcal{F}$*-coalgebra* is defined to be any object U together with a morphism $s : U \longrightarrow \mathcal{F}(U)$. Thus, within the category SET, $[\cdot]$-structures and $[\![\cdot]\!]$-structures are just $\mathscr{P}(\cdot)$-coalgebras and $(\cdot \rightarrow 2)$-coalgebras respectively. Coalgebras form a category: a $\mathcal{F}$*-morphism* between $\mathcal{F}$-coalgebras $\langle U; s\rangle$ and $\langle V; t\rangle$ is given by a morphism $f : U \longrightarrow V$ with a natural commutativity property. As far as $\mathcal{F}$ is covariant, this condition is $t \circ f = f^{\mathcal{F}} \circ s$. In the case where $\mathcal{F}$ is contravariant, it should

[11] This is no longer true for '$\forall x \in x$'. On the other hand, bounded quantifications of the form '$\exists x \in y$' and '$\exists x \in x$' are also preserved since these are actually positive.

be expressed by $f^{\mathcal{F}} \circ t \circ f = s$. It follows that the notion of $\mathscr{P}(\cdot)$-morphism and the one of $(\cdot \to 2)$-morphism differ. What we called a $[\cdot]$-homomorphism is nothing but a $\mathscr{P}(\cdot)$-morphism, which was thus characterized by the condition $[f`v]_{\mathcal{V}} = f``[v]_{\mathcal{U}}$, for every $v \in U$. In these terms, it is easily seen that the corresponding condition for $f$ to be a $(\cdot \to 2)$-morphism would correspond to $f^{-1}``[f`v]_{\mathcal{V}} = [v]_{\mathcal{U}}$, for each $v \in U$, which is clearly a strong requirement for it is equivalent to $u \in_{\mathcal{U}} v \Leftrightarrow f`u \in_{\mathcal{V}} f`v$, for any $u, v \in U$. We might call such a function $f : U \longrightarrow V$ a $[\![\cdot]\!]$*-homomorphism*. When f is injective, this is known as an *embedding*; when f is surjective, it is in particular a $[\cdot]$-homomorphism. Note that it is hopeless to obtain consistency results for formulas preserved under $[\![\cdot]\!]$-homomorphisms since $x \notin x$ is one of these.

2.4 Models for set theory

Given a set-theoretic structure $\mathcal{U}$, $\mathrm{rng}[\cdot]_{\mathcal{U}}$ is nothing but the set of *collectable* subsets of U. We now define $\mathrm{def}[\cdot]_{\mathcal{U}}$ to be the set of *definable* subsets of U, i.e. $\mathrm{def}[\cdot]_{\mathcal{U}} := \{\langle x \mid \varphi(x, \bar{v})\rangle_{\mathcal{U}} \mid \varphi(x, \bar{p}) \text{ in } \mathscr{L}, \bar{v} \text{ in } U\}$. Note that $\mathrm{rng}[\cdot]_{\mathcal{U}} \subseteq \mathrm{def}[\cdot]_{\mathcal{U}}$.

Russell's paradox says that, in any case, $\langle x \mid x \notin x\rangle_{\mathcal{U}} \notin \mathrm{rng}[\cdot]_{\mathcal{U}}$, from which it follows that $\mathrm{rng}[\cdot]_{\mathcal{U}} \subsetneq \mathscr{P}(U)$ (Cantor's theorem). This is not an accident for it can be shown that actually $|\mathrm{def}[\cdot]_{\mathcal{U}} \setminus \mathrm{rng}[\cdot]_{\mathcal{U}}| = |U|$ if U is infinite (see [22]), and, more obviously, that $|\mathscr{P}(U) \setminus \mathrm{rng}[\cdot]_{\mathcal{U}}| > |U|$ unless $|U| = 1$ or 2. So we are very far from the idealistic Fregean situation $U \simeq \mathscr{P}(U) \,/\, U \simeq (U \to 2)$.

In every model $\mathcal{U}$ of any extensional set theory, the set of collectable subsets must have the same size as U. Some set theory might thus be *characterized* in some of its models by the selection of a *specific* class of subsets $\mathcal{F}(U)$ such that $U \simeq \mathcal{F}(U)$. As a famous example, we may quote ZF.

Limitation of size

Given a set U and a cardinal κ, we let $\mathscr{P}_{<\kappa}(U)$ stand in what follows for the set of κ*-finite* subsets of U, i.e. $\{A \subseteq U \mid |A| < \kappa\}$.

Within ZF as framework, it is proved that there exists one and only set U such that $U = \mathscr{P}_{<\aleph_0}(U)$, namely $U = V_\omega$, which is precisely known to be a model of $ZF_{\not\infty}$ (here $[\cdot]_{\mathcal{U}}$ is just taken to be the identity function).[12] And if one wants a model of infinity as well, this is still possible by invoking the

[12] As the notation suggests, $ZF_{\not\infty}$ stands for ZF without the Axiom of Infinity. By the way, we would also remind the reader of the definition of the V_α's, α an ordinal: $V_{\beta+1} := \mathcal{P}(V_\beta)$, for any β, and $V_\lambda := \bigcup\{V_\gamma \mid \gamma < \lambda\}$, if λ is a limit ordinal. Notice that the axioms of ZF^- are just formulated in order that this construction, the so-called *cumulative hierarchy*, can be achieved; in ZF, the universe does coincide with $\bigcup\{V_\alpha \mid \alpha \text{ ordinal}\}$.

existence of a strongly inaccessible cardinal κ, so that V_κ, which is the sole solution in ZF to $U = \mathscr{P}_{<\kappa}(U)$, is itself a model of ZF.

It is worthy of note that if the meta-theory is ZFA, there are different such U's: the minimal one is still V_κ, model of ZF, and the maximal one is now itself a model of ZFA (idem when $\kappa = \aleph_0$ but we lose infinity).

So we have indeed a situation here in which a model for some set theory, which is ZF/ZFA, arises from a bijection - the identity as it is - between a set U and a set of *distinguished* subsets of U, namely the κ-finite subsets. Furthermore, the existence of these *canonical* models perfectly shows that ZF is just the theory of *hereditarily small* and iterative sets; as for ZFA, we simply drop the adjective 'iterative'. It is the reason why the guiding principle of ZF/ZFA for avoidance of the paradoxes is often referred to as the so-called *limitation of size doctrine*.

Adding structure

A natural way of specifying a class of subsets of a given set consists in adding some structure on it and then looking at particular subsets which are defined in terms of the underlying structure. The previous example falls into this in an obvious way: if V_ω is regarded as a topological space with the discrete topology, the finite subsets are just the compact ones, so V_ω now appears as a topological solution $\mathcal{U}$ to $U \simeq \mathscr{P}_{cpact}(U)$, the set of compact subsets of U.

Viewing things that way has at least the merit of suggesting a more interesting situation, in which the underlying topology on U would be such that U itself be compact, so that this latter would be collectable, and so there would be a universal set. In fact, it was shown that such a *compact* solution to $U \simeq \mathscr{P}_{cpact}(U)$ exists.[13] More precisely, it is proved that within a suitable category of T_2-spaces there is one (and only) *compact* solution $\mathcal{U}$ to $U \cong \mathscr{P}_{cl}(U)$, the set of *closed* subsets of U - endowed with a suitable topology. It was called N_ω.

A very characteristic property of the corresponding set-theoretic structure, which is actually shared by any solution $\mathcal{U}$ to $U \simeq \mathscr{P}_{cl}(U)$, is thus the following: every class '$\{x \mid \varphi(x)\}$' can be approximated by a least upper set, the extension of which is just the closure of $\langle x \mid \varphi(x)\rangle_{\mathcal{U}}$ in U. This property is explicitly formulated in Chapter 4 where it is referred to as $(\Diamond)$.

Interestingly, it turns out that the set-theoretic structure associated with N_ω is also a model of $Comp[\varphi(x)]$ for every $\varphi(x)$ in $\mathscr{L}^{[+]}$, which is a *syntactical* fragment of $\mathscr{L}$ - whereas the axioms of ZF are not. It is all the more interesting in that it is shown in [12] that the first-order theory $(\Diamond)+Comp[\mathscr{L}^{[+]}]$,

[13] Within ZFA as framework we may even write $U = \mathscr{P}_{cpact}(U)$.

which was named GPK^+ for historical reasons, can interpret $ZF_{\not\infty}$ in a natural way. And in order to interpret ZF, it suffices to add to GPK^+ an axiom ensuring the existence of the least infinite Von Neumann ordinal ω, which yields the theory GPK^+_∞ deeply investigated in [12]. Note that it was originally proved in [17] that with a large cardinal assumption - namely the existence of a *weakly compact* cardinal - the technique used to construct N_ω can be so carried out as to fulfil that relevant axiom of infinity.

Topological solutions

In light of what has been said, we may define a *topological model* for set theory to be any set-theoretic structure $\mathcal{U}$ that is solution to an equation $U \simeq \mathcal{F}(U)$, or even better $U \cong \mathcal{F}(U)$, where $\mathcal{F}(\cdot)$ is some power-object/exponential acting on a category of topological spaces. Typically, one may think of $\mathcal{F}(U)$ to be $\mathscr{P}_{cl}(U)$, the set of *closed* subsets of U, or $\mathscr{P}_{op}(U)$, the set of *open* ones, both endowed with some suitable topology if required. It is worth emphasizing that by a solution $\mathcal{U}$ we really mean a topological space U *together with* a bijection / homeomorphism '$[\cdot]_{\mathcal{U}}$' realizing $U \simeq \mathcal{F}(U) \,/\, U \cong \mathcal{F}(U)$.

In such set-theoretic structures it might be said that it is the *indiscernibility* associated with the underlying topology that governs the collecting process. And that such equations can be solved within suitable categories of topological spaces would thus mean that, to some extent, the Fregean problem is solvable under indiscernibility.

As we shall see, however, we are still far from the idealistic Fregean perspective. Indeed, in most cases the axiomatic set theory to which topological models give rise is rather insignificant in itself from the foundational point of view. In fact, the sole exception we know is our guiding example above.[14]

Anyway, we are not seeking a new proposal in this thesis. We are only interested in consistency results on comprehension/abstraction restricted to some *syntactical* fragments of $\mathscr{L} / \mathscr{L}_\tau$. The next chapter is precisely intended to show how topological considerations then naturally come into the picture.

[14] It is shown in [12] that GPK^+_∞ is as strong as 'Kelley-Morse + On is ramifiable'.

Chapter 3

CONTINUITY: A SAFETY PROPERTY

Here we make a survey of various proposals which are closely related to our investigations. The content of this chapter is to appear in [29].

3.1 Deviation in logic

It must be admitted that mathematical investigations in providing *alternative* semantics have carried innovative ideas, and if all have not led to further developments and applications, they have often led to a better understanding of the topic considered.

Even within a well-established framework, the use of alternative semantics has proved its fruitfulness. As an example, for independence results in ZF, one may quote the Boolean-valued version of forcing due to Scott and Solovay, in which a set is conceived of as a function that takes its values into a given complete Boolean algebra, no more into the 2-valued one. This concerns classical logic and perhaps would remind the reader of the primal use of many-valued semantics for proving the independence of axioms in propositional logic. Note that there is no need to be interested in any possible meaning of the additional 'truth values' to do that. We would rather say that the explanation is in the application.

Now, it is legitimate to enquire whether the use of many-valued semantics and the like could not also benefit, in one way or another, our understanding of the set-theoretical paradoxes. After all, to know which *logical systems* whatsoever can support the full comprehension scheme, or some fragments of it, is an interesting question in itself, at least not devoid of mathematical interest. We would then let the various proposals speak for themselves.

As said in Chapter 1, we shall be concerned with type-free approaches in this thesis, and mainly with those that somehow reject classical negation.[1]

[1] For a solution route in which the definition of $\equiv$ is altered, while classical negation is

So we review in this chapter some attempts in that direction, giving an historical account on the subject in connection with the pioneering work of Skolem in Łukasiewicz's logics. Our aim is to trace and stress the use of *fixpoint arguments* in semantic consistency proofs and, thereby, the role of *continuity* in avoiding the paradoxes. It will then become apparent how close these investigations were to other contemporary ones, as Kripke's seminal work on the liar paradox and Scott's on models for the untyped λ-calculus. We also aim to show how the distinction comprehension/abstraction and the existence of topological models - both of which being explored in this thesis - have emerged from such 'deviant' proposals.

3.2 Moh Shaw-Kwei's paradox

The existence of the Russell set is prohibited in classical logic. By tampering with the negation, some alternative logics have proved more tolerant. Nevertheless, there exist other sets that can affect such non-classical systems. This was illustrated in 1954 by Moh Shaw-Kwei [32] who presented the following extended version of Curry's paradox.

Let $\rightarrow$ be the *official* implication connective of the logic considered. Precisely, in order that $\rightarrow$ may be referred to as an implication connective, it is assumed that *modus ponens* holds, namely

$$MP: \quad \varphi, \varphi \rightarrow \psi \vdash \psi$$

where $\vdash$ is the consequence relation associated with the logic.

To express Moh's paradox, we define the *n-derivative* $\rightarrow^n$ of the implication inductively as follows:

$$\varphi \rightarrow^0 \psi :\equiv \psi \qquad \text{and} \qquad \varphi \rightarrow^{n+1} \psi :\equiv \varphi \rightarrow (\varphi \rightarrow^n \psi) \qquad (n \in \mathbb{N}).$$

Then the implication connective is said to be *n-absorptive* if it satisfies the *absorption rule of order n*, that is,

$$A_n: \quad \varphi \rightarrow^{n+1} \psi \ \vdash \ \varphi \rightarrow^n \psi.$$

Now, assuming that the implication is n-absorptive for some $n > 0$, it is easily seen that any formula can be derived from the existence of the set $C_n :=$ '$\{x \mid x \in x \rightarrow^n \bot\}$', where $\bot$ is a 'falsum' constant defined in such a way that $\bot \vdash \varphi$, for any φ.

maintained, the reader is referred to [3].

Proof.

$$
\begin{array}{rlll}
Comp[\mathscr{L}] & \vdash \; \forall x\,(\, x \in C_n \leftrightarrow (x \in x \to^n \bot)\,) & \text{[Comprehension]} & (1)\\
& \vdash \; C_n \in C_n \leftrightarrow (C_n \in C_n \to^n \bot) & \text{[Univ. Quant. Elimin.]} & (2)\\
& \vdash \; C_n \in C_n \to^n \bot & [(2)^{\rightarrow}, A_n] & (3)\\
& \vdash \; C_n \in C_n & [(2)^{\leftarrow}, (3), MP] & (4)\\
& \vdash \; \bot & [(3), (4), MP(n \text{ times})] & (5).
\end{array}
$$

Thus any formula can be derived and the theory is meaningless. ⊣

As a particular case we have C_1 = '$\{x \mid x \in x \to \bot\}$', which might be called the Curry set, and then we recover the Russell set R = '$\{x \mid x \notin x\}$' if $\neg\varphi$ is defined by $\varphi \to \bot$, as it is the case in classical logic. Note incidentally that C_0 = '$\{x \mid \bot\}$', but this later is not problematic, of course.

3.3 The Łukasiewicz logics

A nice illustration is supplied by the most popular many-valued logics, the Łukasiewicz ones. We shall content ourselves here with recalling the truth-functional characterization of the connectives and quantifiers of these logics.

The set of truth values for the *infinite*-valued Łukasiewicz logic $Ł_\infty$ is taken to be the real unit interval $I := [0, 1] \subseteq \mathbb{R}$ with its natural ordering, which will be referred to as the *truth ordering* and denoted by $\leqslant_T$. The sole *designated* value is 1. Here are the truth functions of the logical operators:[2]

- $\bot$ is 0 and $\top$ is 1;
- negation is defined by $\neg x := 1 - x$, for any $x \in I$;
- conjunction and disjunction are the minimum and maximum w.r.t. $\leqslant_T$, i.e., $x \wedge y := \min_{\leqslant_T}\{x, y\}$ and $x \vee y := \max_{\leqslant_T}\{x, y\}$, for any $x, y \in I$;
- quantifiers are thought of as generalized conjunction and disjunction, i.e., for any $A \subseteq I$, $\forall A := \inf_{\leqslant_T} A$ and $\exists A := \sup_{\leqslant_T} A$;
- last but not least, the truth function of the implication is specifically defined by $x \to y := \min_{\leqslant_T}\{1, 1 - x + y\}$. Notice that this is not the 'material conditional' $x \supset y := \neg\, x \vee y$; we only have $x \supset y \leqslant_T x \to y$. Thus defined, $\to$ is a characteristic function of the truth ordering, for we have $x \to y = 1$ if and only if $x \leqslant_T y$, which $\supset$ fails to fulfil here. Consequently, $\equiv$, which is just taken to be $\leftrightarrow$, is still characterized by $x \equiv y = 1$ if and only if $x = y$. That is to say, $\mathcal{U} \models \varphi \equiv \psi$ if and only if $|\varphi|_{\mathcal{U}} = |\psi|_{\mathcal{U}}$, as in classical logic.

[2] We use the same notation for the operators and their truth functions.

For our purposes, we remark that if we equip I with the usual topology of the real line, then each of the truth functions of the connectives is continuous. The truth functions of the quantifiers are continuous as well with respect to a reasonable topology on the set of subsets of I. This *extra-logical* property of the interpretation of the logical operators will be of interest to us.

The n-valued Łukasiewicz logic $Ł_n$ ($n \geqslant 2$) is obtained by restricting the set of truth values to $I_n := \{\frac{k}{n-1} \mid k \in \mathbb{Z}_n\}$. Particular cases are then the 2-valued logic $Ł_2$, which is nothing but classical logic, and the 3-valued one $Ł_3$, which historically was the first many-valued logic introduced by Łukasiewicz.

It was noticed in [32] that the implication is $(n-1)$-absorptive in $Ł_n$, whereas it is not n-absorptive in $Ł_\infty$, for any value of n, whereupon the author asked whether one could develop the naive theory of sets from $Ł_\infty$.

3.4 Skolem's conjecture

This observation was the starting point in the late fifties of a course of papers initiated by Skolem, who conjectured and tried to prove in [35] the consistency of the full comprehension scheme in $Ł_\infty$.[3] On the way, Skolem was led to considering and investigating the consistency problem of some fragments of that scheme in $Ł_3$ & $Ł_2$ [36, 37, 38], on which we are going to dwell later.

Skolem's conjecture was partially confirmed by Skolem himself in [35] and by Chang and Fenstad in different papers [10] & [15]. For instance, Skolem only showed the consistency of the comprehension scheme restricted to formulas containing no quantifiers, while Chang had quantifiers but no parameters, or parameters but then some restrictions on quantifiers.

From the technical point of view, what should be said is that all these first attempts are *semantic* and that their proofs are based on the original method of Skolem, using at some stage a fixpoint theorem, namely *Brouwer's fixpoint theorem* for the space I^m, $m \in \mathbb{N}$ (or even for $I^{\mathbb{N}}$), which states that any *continuous* function on I^m (or $I^{\mathbb{N}}$) has a fixpoint. We will meet another famous fixpoint theorem which was also used, but rather implicitly, in Skolem's papers [36, 37].

We shall show in 3.6 how fixpoint arguments can be involved in such semantic consistency proofs. Let us first see what a many-valued set-theoretic structure looks like.

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[3] As just mentioned, this suggestion was made by Moh Shaw-Kwei in [32]. It should be remarked however that Skolem does not cite that work. He might have arrived at his conclusions independently.

3.5 Many-valued structures

A set-theoretic structure $\mathcal{U}$ for a many-valued logic of which T is the set of truth values is defined like the 2-valued classical version(s) presented in 2.3:

$$\langle U; \epsilon_{\mathcal{U}} \rangle \quad \text{where} \quad \epsilon_{\mathcal{U}} : \ U \times U \longrightarrow T$$

or

$$\langle U; [\![\cdot]\!]_{\mathcal{U}} \rangle \quad \text{where} \quad [\![\cdot]\!]_{\mathcal{U}} : \begin{array}{l} U \longrightarrow (U \to T) \\ v \longmapsto (u \mapsto \epsilon_{\mathcal{U}}`(u, v)). \end{array}$$

In the case of $Ł_2$, we noticed in 2.3 that a *normal* structure $\mathcal{U}$ is *extensional* if and only if $[\![\cdot]\!]_{\mathcal{U}}$ is injective. But in the case of $Ł_\infty \,/\, Ł_n$, $n > 2$, what it is for a structure to be *extensional* is not so clear, for the principle of extensionality itself may be subject to different interpretations. This is because in $Ł_\infty \,/\, Ł_n$, $n > 2$, the implication $\to$ is no longer the translation at the object-language level of the consequence relation $\vdash$ (only Modus Ponens remains). Therefore, depending on whether it is considered as a rule or as an axiom, and depending also on whether equality is taken as primitive or not, different versions of extensionality are conceivable (note that '$\sharp$' has no effect in $Ł_2$):

$$^\flat Ext : \ x \mathrel{\underset{\cdot}{=}} y \vdash x \doteq y \qquad\qquad ^\flat Ext^\sharp : \vdash \forall x \forall y (x \mathrel{\underset{\cdot}{=}} y \to x \doteq y)$$

$$Ext : \ x \mathrel{\underset{\cdot}{=}} y \vdash x = y \qquad\qquad Ext^\sharp : \vdash \forall x \forall y (x \mathrel{\underset{\cdot}{=}} y \to x = y).$$

In order for $=$ to bear the title of *equality*, it is to be required that it be so interpreted in any structure as to guarantee the principle of *substitutivity* (again, either as a rule or as an axiom scheme). This can be met by simply defining the truth function of $=$ on any structure by $=_{\mathcal{U}}`(u, v) := 1$ if and only if $u = v$ in $U$, and $=_{\mathcal{U}}`(u, v) := 0$ otherwise. Incidentally, it was noticed in [10] that this *strict* interpretation will never yield a model of $Comp[\mathscr{L}] + Ext^\sharp$. A more reasonable definition of the equality relation in a structure may be any one such that $=_{\mathcal{U}}`(u, v) = 1$ if and only if $u = v$ in U, and that is all, for this suffices to ensure that $=$ has the substitutivity property (but as a rule, not as an axiom scheme actually). Any structure equipped with such an interpretation of $=$ will be said to be *normal*. In that case, it is easy to see that $[\![\cdot]\!]_{\mathcal{U}}$ is injective if and only if $\mathcal{U}$ is extensional in the sense of Ext. This latter shall henceforth be our preferred version of extensionality.

Accordingly, the universe U of any extensional normal structure $\mathcal{U}$ may be identified with a (proper) subset of $(U \to T)$, namely the range of $[\![\cdot]\!]_{\mathcal{U}}$, so that $\mathcal{U}$ now appears as a solution to $U \simeq [U \to T]$, where $[U \to T]$ is a suitable set of functions $U \longrightarrow T$, or say, a set of *suitable functions* $U \longrightarrow T$.

Of course, because of Cantor's theorem, not every function $U \longrightarrow T$ is suitable. Worse, even some simple propositional truth-functions may not be. As we shall now see, only truth-functions having fixpoints are welcome.

3.6 The fixpoint property

Let $A(\cdot)$ be any propositional function in one variable, and let $\alpha : T \longrightarrow T$ denote its truth-function. Now suppose we are given a set-theoretic structure $\mathcal{U}$ in which '$\{x \mid A(x \in x)\}$' has an interpretation, say $a \in U$. Then we must have $|a \in a|_{\mathcal{U}} = |A(a \in a)|_{\mathcal{U}} = \alpha(|a \in a|_{\mathcal{U}})$, showing that $\alpha : T \longrightarrow T$ has a fixpoint, namely $|a \in a|_{\mathcal{U}}$. We shall refer to this as *the fixpoint property*. It follows therefrom that if ever $\mathcal{U} \models Comp[\mathscr{L}]$, then *any* propositional truth-function $\alpha : T \rightarrow T$ must have the fixpoint property.

In the case of $Ł_n$, $n \geqslant 2$, not every propositional truth-function can have the fixpoint property. An example was actually provided in the proof of Moh Shaw-Kwei's paradox. Indeed, just take $\alpha(x) := x \rightarrow^n \bot$. A simple computation shows that $\alpha(x) = \min_{\leqslant_T}\{1\,,\, n(1-x)\}$, from which it is easily seen that α has no fixpoint on I_n.

On the other hand, in the case of $Ł_\infty$, the fixpoint property is fulfilled. For we noticed in 3.3 that the truth-functions of the logical connectives and quantifiers of $Ł_\infty$ are continuous, and then so is any truth-function $\alpha : I \longrightarrow I$ defined from these. Now, by Brouwer's theorem, any such α has a fixpoint. It is then not surprising that the Brouwer fixpoint theorem was involved in the first attempts to provide a model of $Comp[\mathscr{L}]$ in $Ł_\infty$. Broadly, this example would even suggest that *continuity*, in a very comprehensive manner as we shall see, might be regarded as a *safety* property against the paradoxes.

3.7 White's solution

The use of an abstractor within a many-valued framework is conceivable too. And it was precisely by using an abstraction operator, and by a *proof-theoretic* method in fact, that Skolem's conjecture was finally established much later by White in 1979 ; see [41]. What was actually proved therein is the consistency of $Abst[\mathscr{L}_{\tau*}]$ in $Ł_\infty$. It was noticed by the author himself that his system is too weak in order to develop classical first-order number theory inside. He also showed that ${}^\flat Ext^\sharp$ cannot be consistently added, but we ignore whether ${}^\flat Ext$ can be. In fact, it is still not known whether $Comp[\mathscr{L}] + Ext$ is consistent. If so, any semantic proof showing the existence of *natural* models would be welcome in that such models would give rise to a full universe of *fuzzy* sets.[4]

We now leave the consistency problem of the full comprehension scheme in $Ł_\infty$, and concentrate on the one of some syntactical fragments of it in $Ł_3$ and $Ł_2$, as initiated by Skolem in [36, 37, 38]. Although the use of set abstracts in the language was again salutary to prove their consistency, we are going to see that it can also be fatal to extensionality in the presence of equality in formulas defining sets.

3.8 Abstraction and extensionality

Assuming extensionality, set abstracts can be eliminated. But in doing that, some undesirable connectives may sneak in by the back door. This is particularly well illustrated by the rule $R2$ stated in 2.1:

$$\tau = `\{x \mid \varphi(x)\}' \quad \Leftrightarrow \quad \forall z(z \in \tau \leftrightarrow \varphi(z))\,,$$

the following instance of which is still more eloquent

$$`\{x \mid \bot\}' = `\{x \mid \varphi(x)\}' \;\Leftrightarrow\; \forall z(\bot \leftrightarrow \varphi(z)) \;\Leftrightarrow\; \neg\varphi \qquad (\dagger).$$

Now it is easily proved that, assuming Ext, the Russell set can be defined in $Ł_2$ without negations or implications, simply by using set abstracts and equality in the language. This results in the key observation that follows:

Fact 3.8.1. $Abst[\mathscr{L}_\tau{}^+]$ is incompatible with Ext

Proof. Assume $Abst[\mathscr{L}_\tau{}^+]$ and define $r := \{x \mid \{z \mid x \in x\} = \{z \mid \bot\}\}$. Assuming Ext, it follows from (†) that $r = `\{x \mid x \notin x\}'$. ⊣

This sort of 'paradoxes' first appeared in Gilmore's work on partial set theory [19].[5] It should be remarked that the Russell set is no longer contradictory in Gilmore's set theory. What was showed in [19] is that, within an extensional universe, a substitute for it can be defined by using set abstracts and equality in the language, in a similar (yet more subtle) manner as above.

Although his motivations were elsewhere, that work by Gilmore could have equally been expressed within the 3-valued Łukasiewicz logic. As a matter of fact, Brady in [8] directly adapted Gilmore's technique to $Ł_3$ to

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[4] So far we have only obtained partial results, comparable to those of [10] & [15], by using techniques described in [4]. This should be discussed elsewhere.

[5] It is worthy of note that Gilmore's work was first publicized in 1967, but the inconsistency of extensionality only appeared in 1974.

much strengthen Skolem's initial result, showing the consistency of an abstraction scheme in that logic, but without equality in the language (as in Skolem). By this mere fact, however, a significant departure in [8] is that the author succeeded in proving that his model is extensional, and thus, though he was not aware of that,[6] he actually proved a complementary result of Gilmore's. Namely, one can recover extensionality by dropping equality out of formulas defining sets.

To clearly express those results of Gilmore and Brady, let $\mathscr{L}_\tau^{\not\to}$ stand for the set of $\mathscr{L}_\tau$-formulas containing no occurrences of $\to$. Notice that this restriction is meaningful in view of Moh Shaw-Kwei's paradox. Then we have

Theorem ([19]).
$Abst[\mathscr{L}_\tau^{\not\to}]$ is consistent in $Ł_3$, but inconsistent together with Ext.

Theorem ([8]).
$Abst[\mathscr{L}_{\tau *}^{\not\to}] + {}^\flat Ext$ is consistent in $Ł_3$.

We mention here that similar results apply as well to the *paraconsistent* counterpart of $Ł_3$, the quasi-relevant logic RM_3 (e.g. in [9]). For a survey of the paraconsistent approach, we refer the reader to [27].

Before Gilmore/Brady's method, Skolem had built extensional models of a comprehension scheme in $Ł_3$ restricted to formulas not containing any occurrence of $\to$, but not any quantifier either (and without equality in the language). In fact, his technique of proof in [36, 37] cannot be extended in order to handle quantifiers. But Skolem showed in [37] that it can be adapted to $Ł_2$, initiating by the way, as far as we know, the consistency problem for *positive* comprehension principles. Moreover, he presented different techniques in [37] and [38], one of which finally led him to prove the consistency of $Comp[\mathscr{L}_*^+] + Ext$ (see [38], Theorem 1). The model he discovered is going to be further analyzed in Chapter 5, where all its secrets will be revealed.

Without any references to Skolem, the consistency problem for positive comprehension principles was reinvestigated and invigorated much later in the eighties, where it would seem to have his source in Gilmore's work precisely; see [21] & [17]. Then it is not surprising that similar results to those stated for $Ł_3$ could be proved for $Ł_2$.

Theorem ([21]).
$Abst[\mathscr{L}_\tau^{+}]$ is consistent in $Ł_2$, but inconsistent together with Ext.

Theorem ([24]).
$Abst[\mathscr{L}_{\tau *}^{+}] + {}^\flat Ext$ is consistent in $Ł_2$.

[6] Cf. previous footnote.

But here much more interesting *extensional* models were actually discovered in order to recover equality in formulas defining sets, and thus by giving up the use of set abstracts.

Theorem ([17]).
$Comp[\mathscr{L}^+] + Ext$ is consistent in $Ł_2$.

We also mention that the technique used to construct them was subsequently adapted by Hinnion in [23] to the partial and the paraconsistent cases with different success (see also [28, 27] for the paraconsistent version).

The rest of this chapter is in a sense devoted to showing how such models can be obtained. To proceed, we first have to point out the guiding idea, which was already subjacent in the original work of Skolem.

3.9 A particular case of continuity

The key step in Skolem's proofs of the consistency of a comprehension scheme in $Ł_3$ and $Ł_2$ (see [36] and [37]) is again the observation that the truth functions of formulas defining sets have the fixpoint property. Of course, as the set of truth values is discrete, it is no longer possible to invoke Brouwer's theorem to see that. As a matter of fact, Skolem contents himself in [36] with noticing that there are exactly eleven propositional truth-functions in one variable constructible in $Ł_3$ without using $\rightarrow$, and that each of them really has a fixpoint. In [37] a similar remark for positive formulas is applied to $Ł_2$. Although this was not noticed by Skolem, it is another famous fixpoint theorem that is hidden behind these observations, namely the *Knaster-Tarski theorem* for ordered sets, on which we shall now dwell.

We would remind the reader that a particular case of continuity is *monotonicity.* For it is well known that if any (partially) ordered set is endowed with the topology for which the open subsets are just the upper sets, i.e., A is *open* if and only if $x \geqslant a \in A \Rightarrow x \in A$, then the continuous functions are exactly the monotone ones. This technically convenient topology is referred to as the *Alexandroff topology.* It may be defined on any *preordered* set actually, and it is easily characterizable as we shall see later in 4.7.

Now it turns out that the ordered sets on which any continuous/monotone function has a fixpoint are identifiable. These are the *dcpo*'s.

We say that an ordered set U is *directed complete*, or a *dcpo* for short, if any directed subset of U has a least upper bound, denoted by $\bigvee A$; where $D \subseteq U$ is said to be *directed* if $D \neq \emptyset$ and for any $a, b \in D$, there exists $c \in D$ with $a, b \leqslant c$. If in addition $\bigvee \emptyset$ exists, that is to say, if U has a least element, then we say that U is a *pointed* dcpo (some authors use the term *cpo*).

Any complete ordered set is obviously a pointed dcpo, and the relevance of this notion lies in the following important theorem, which is in fact the best refinement of Tarski's original version for complete lattices:

Theorem (Knaster-Tarski).
Let U be a pointed dcpo and let $f : U \longrightarrow U$ be monotone. Then $Fix(f)$, the set of all fixpoints of f, is itself a pointed dcpo. Consequently, f has a (least) fixpoint. Moreover, if U is a complete lattice, so is $Fix(f)$.

We denote the least fixpoint of f by $\mu(f)$. Using the machinery of ordinal numbers, $\mu(f)$ can be reached inductively by iterating f from the least element of U (whereas the existence of maximal ones relies on Zorn's lemma, unless U is a complete lattice or is finite, of course).

To complement the Knaster-Tarski theorem, it has been shown that *a (pointed) ordered set on which any monotone function has a fixpoint is necessarily a dcpo.* This is, by far, much more difficult to prove (see [1] for references).

Now, we remark that any of the I_n's or I, with the *truth* ordering $\leqslant_T$, is a pointed dcpo; and it is easily seen that all connectives and quantifiers *except* negation *and* implication are monotone - whereas we would remind the reader that *all* were continuous with respect to the *usual* topology on I. Of course, if both negation and implication are rejected, there is absolutely no need to add some imaginary truth-values, so this only makes sense for $Ł_2$, which incidentally is the strongest logic in the family.

In the case of $Ł_3$, however, the set of truth degrees may be equipped with another ordering, the so-called *knowledge/information* ordering $\leqslant_K$, which comes from the various attempts to explain the middle value $\frac{1}{2}$ as '*unknown*', '*undefined*', '*undetermined*', '*possible*', or whatever expressing in some sense a *lack* of truth value. Pictorially:

$$\begin{array}{ccccc} 0 = \{\texttt{false}\} & & & & \{\texttt{true}\} = 1 \\ & \diagdown & & \diagup & \\ & & \frac{1}{2} = \{\,\} & & \end{array}$$

The partially ordered set '$\diagdown\!\diagup$' thus defined is actually the simplest example of a pointed dcpo that is not a chain. With respect to this ordering, it is readily seen that all connectives and quantifiers except implication *only* are monotone. It follows from the Knaster-Tarski fixpoint theorem that any truth-function that is not defined by means of $\rightarrow$ has the fixpoint property. Thus we may trace back the embryonic use of this theorem in semantic consistency proofs to Skolem's papers.

3.10 Kripke-style models

Interestingly it is again the same theorem that is implicitly invoked, but at another level, in Gilmore and Brady's work in providing their *term models*.

Roughly, the universe of these models is fixed in advance and made of set abstracts, regarded as *syntactical* expressions of the form $\{x \mid \varphi\}$ for suitable formulas φ (e.g. $\mathscr{L}^{+}_{\tau*}$, $\mathscr{L}^{\not\to}_{\tau*}$); and then, by a fixpoint argument, the membership relation is determined inductively in such a way that $\{x \mid \varphi\}$ be a solution to the φ-instance of the *abstraction scheme* under consideration.

This so-called *inductive method* was later popularized by Kripke [25] in his work on the liar paradox. For a comprehensive description and analysis of the connection between these works, we may refer the reader to [14].

We will see the inductive method in action in Chapter 6 where we review and complete the results in [24].

Let us here sing the praises of another useful technique which has its source in Scott's seminal work on the λ-calculus. As anyway we will need some basic facts about dcpo's in this thesis, we first supply the reader with a few prerequisites, referring systematically to [1] for more complete treatments.

3.11 Addendum on dcpo's

Dealing with dcpo's, it is natural to be concerned with those monotone functions that preserve suprema of *directed* subsets. Let U, V be dcpo's. We say that a function $f : U \longrightarrow V$ is *Scott-continuous* if and only if f is monotone and for all *directed* subset $D \subseteq U$, we have $f`\bigvee D = \bigvee f``D$. (Note that Scott-continuous functions need not preserve least elements when they exist.) The relevance of this notion lies in the fact that, for a Scott-continuous function f on a pointed dcpo, ω steps (at most) are enough to reach $\mu(f)$.

We write $\langle U \to V\rangle$ for the set of all Scott-continuous functions $U \longrightarrow V$. It is easy to see that $\langle U \to V\rangle$, equipped with the pointwise ordering, is itself a dcpo, and then that $\mu : \langle U \to U\rangle \longrightarrow U$ is Scott-continuous. We let *DCPO* stand for the category of dcpo's with Scott-continuous functions as morphisms, and we refer the reader to [1] for any property of *DCPO* we shall mention and use without proof. Notice in particular that it is shown in [1] that *DCPO* is cartesian closed, the exponential of which being just $\langle \cdot \to \cdot\rangle$.

The appropriate topology for a dcpo is called the *Scott topology*; this is coarser than the Alexandroff one. Given a dcpo U, we say that $A \subseteq U$ is Scott-open if and only if A is an upper set and for any directed set $D \subseteq U$, $\bigvee D \in A \Rightarrow D \cap A \neq \emptyset$; and thus we say that $B \subseteq U$ is Scott-closed if and only if B is a lower set and for any directed set $D \subseteq U$, $D \subseteq B \Rightarrow \bigvee D \in B$. It is now easily seen that the Scott-topologically-continuous functions are exactly the Scott-continuous ones as defined above. It is also apparent that $A \subseteq U$ is Scott-open if and only if $\imath_U‘A : U \longrightarrow 2$ is Scott-continuous, where 2 is equipped with its natural truth ordering (i.e. $0 < 1$). Likewise, one can see that $B \subseteq U$ is Scott-closed if and only if $\imath_U‘B : U \longrightarrow 2^*$ is Scott-continuous, where 2^* is the order dual of 2 (i.e. $1 < 0$). Obviously $2 \cong 2^*$, but this will result in some schizophrenia, as we shall see in Chapter 5 notably.

3.12 Scott-style models

In providing models for the *untyped* λ-calculus, Scott discovered that the Knaster-Tarski theorem is reflected within suitable subcategories of *DCPO*.

Roughly, it was proved that for a wide variety of functors $\mathcal{F}(\cdot)$ acting on those categories, the reflective equation $U \cong \mathcal{F}(U)$ has a least solution (obtained by *inverse limit*). Such fixpoints have naturally proposed themselves as *semantic domains* of programming languages, so the mathematical branch in theoretic computer science that investigates this is called *domain theory*. We again rely on [1] for proofs and further motivation.

Let T stand in what follows for $I_2\,(= 2)$ with the truth ordering $\leqslant_T$, or for I_3 with the knowledge ordering $\leqslant_K$, both of which, we recall, are dcpo's.

It is very tempting, by using techniques of domain theory, to try to solve a reflexive equation of the form $U \cong \mathcal{F}(U) \subseteq [U \to T]$, where this latter is the set of all *monotone* functions $U \longrightarrow T$ (which is a dcpo as well). Any fixpoint solution $\mathcal{U}$ to such an equation will thus give rise to a set-theoretic universe in which sets are conceived of as some special but *monotone* functions $U \longrightarrow T$. Therefore, by virtue of the monotonicity of the connectives and quantifiers, and for a suitable choice of $\mathcal{F}(\cdot)$, such a $\mathcal{U}$ may seem to be a good candidate for an extensional normal model of $Comp[\mathscr{L}^+]$ when T is $\langle I_2; \leqslant_T\rangle$, or of $Comp[\mathscr{L}^{\not\to}]$ when T is $\langle I_3; \leqslant_T\rangle$.

However attractive this idea is, it will not enable us to recover equality in formulas defining sets. This is because $=_\mathcal{U}$ is not monotone on a normal structure $\mathcal{U}$. Let us illustrate this with the two-valued case. We may assume that U is a pointed dcpo and that $|U| > 1$, so that there do exist *distinct* u, v in U such that $u \leqslant_U v$.[7] Now, if $=_\mathcal{U} : U \times U \longrightarrow 2$ was monotone, then we should have $=_\mathcal{U}‘(u,u) \leqslant_T =_\mathcal{U}‘(u,v)$, which is impossible since $=_\mathcal{U}‘(u,u) = 1$ and $=_\mathcal{U}‘(u,v) = 0$. A similar argument applies to the 3-valued case.

[7] This follows from the fact that $\mathcal{U}$ is a solution in *DCPO* to $U \cong \mathcal{F}(U) \subseteq [U \to T]$.

Equality is still missing, but the idea remains appealing in that it could then serve to provide *natural* models of $Comp[\mathscr{L}_*^+]$ or $Comp[\mathscr{L}_*^{\not\to}]$.

Surprisingly it turns out that the least solution $\mathcal{U}$ to $U \cong \langle U \to 2\rangle$ is just one of the twin models of $Comp[\mathscr{L}_*^+]$ given by Skolem in [37, 38]. This is going to be proved in Chapter 5, where it is further shown that $\mathcal{U}$ is actually a model of $Abst_{\circlearrowleft}[\mathscr{L}_{\tau *}^+] + Ext$. The other twin model can be obtained as the least solution to $U \cong \langle U \to 2^*\rangle$. Of course, these are isomorphic as dcpo's (because $2 \cong 2^*$), but not as set-theoretic structures (for the Scott-isomorphisms $[\cdot]_{\mathcal{U}}$ differ). As a matter of fact, topologically speaking, in the one the collectable subsets are just the Scott-*open* subsets, whereas in the other these are the Scott-*closed* ones, whereby we thus establish in Chapter 5 the existence of *topological* models of positive *abstraction*.

Such an attempt for the 3-valued case is largely explored in [5], but with little success. Nevertheless, the authors finally show that the structure so constructed contains a model of '*rough set theory*' within its maximal elements (see also [6]). As we shall see in Section 4.7, this is another (very different) example of topological model actually.

It should also be remarked that, curiously, the 3-valued paraconsistent approach has proved more successful than the 3-valued paracomplete one. We may again refer the reader to [27] for a sketchy analysis of that asymmetry.

3.13 The solution

Topological considerations have thereby come on. And now we have some examples of topological models $\mathcal{U}$ satisfying $U \cong \mathscr{P}_{op}(U) \,/\, U \cong \mathscr{P}_{cl}(U)$, the question as to which of these, if any, may suit our purposes best is legitimate.

Clearly, if our goal is to recover identity in formulas defining sets, only those that are solution to $U \cong \mathscr{P}_{cl}(U)$ are worth being sought after, but then within categories of, at least, T_1-spaces, which dcpo's are not.

As any deep piece of work, Scott's has been a source of inspiration. Thus, it was shown in [4] that another famous fixpoint theorem involving *continuous* functions is also reflected within some suitable category. This theorem is *Banach's* for *contracting* functions on *complete metric spaces*, *cms* for short, which too have been successfully used in modelling programming processes.[8]

Now, by using the technique described in [4], it is proved that the reflexive equation $U \cong \mathscr{P}_{cl}(U)$ has a unique (up to isomorphism) solution among cms's, which are T_2-spaces. This solution is just the space N_ω we mentioned

[8] Recently, the framework of so-called '*continuity spaces*', a common refinement of partially ordered sets and metric spaces, has been proposed to develop a general theory of semantics domains. The interested reader is referred to [16].

in Section 2.4, which, as said, turns out to be a model of $Comp[\mathscr{L}^{[+]}] + Ext$. The equality relation makes hereby its entrance in formulas defining sets within an extensional universe; and as in $Ł_2$ this entrance coincides with the one of $\dot{=}$, some *bounded* quantifications had anyhow to come along with. It is then not surprising that N_ω fulfils $Comp[\mathscr{L}^{[+]}]$, and not only $Comp[\mathscr{L}^{+}]$. Besides, as stressed in Section 2.3, $\mathscr{L}^{[+]}$ is not an unnatural class of formulas.

We thus have related the emergence of a natural *topological* solution to 'the consistency problem for positive comprehension principles', which, as we saw, originated in the use of 'deviant' logics. Note that N_ω is actually the simplest member of a family of topological solutions whose existence was established in [17] (relying on some large cardinal assumptions, and without any reference to [4]). These structures, subsequently called *hyperuniverses*, have extensively been studied by Forti, Honsell and Lenisa in several papers, e.g. in [18], where the authors even proposed hyperuniverses as an universal framework for investigating the semantics of programming languages.

In such topological models the collectable subsets are just the closed ones. Thus, even though not every class can define a set, each class in such models may, at least, be optimally *approximated* by the smallest set containing it. This alternative proposal was the core of Weydert's thesis [40], in which the author was independently led to proving the existence of hyperuniverses. Note that the constructions given in [17] and [40] were actually both inspired by the pioneer work of Malitz [30]; see [17] for an historical account.

Such an idea of approximation of the full comprehension scheme was originally considered by Skala in [34], and then refined by Manakos in [31].[9] Although semantic proofs of the consistency of (some extensions of) Skala's theory appeared in [33], [39] and [20], it seems that no *topological* attempt to characterize the models has been made. So in the next chapter we revisit Skala's topological set theory on both the axiomatic and semantic sides.

[9] Skala's paper is cited in the references of [40], and even in those of [30], but with no further comment.

Chapter 4

ON TOPOLOGICAL SET THEORY

We review and resume the results in [34] & [31] so as to give new insights into them. In most cases the proof is just routine but is not omitted for all that. Our most significant results are Theorem 4.5.4 - a negative one - and Theorem 4.6.5 - a rather positive one.[1] Both follow from a straightforward topological characterization of the models of the theories considered in [34] & [31]. This seems not to have been noticed before. The content of this chapter is recapitulated in [13]. Note that Extensionality is tacitly assumed throughout this chapter, and that a set-theoretic structure $\mathcal{U}$ is looked at as $\langle U; [\cdot]_{\mathcal{U}} \rangle$. It is also worth remembering the abbreviations given in Section 2.1.

4.1 The closure scheme

We are interested here in any *extensional* set theory based upon the following *approximation scheme*:

$$(\Diamond): \quad \left| \begin{array}{l} \text{For every formula } \varphi(x), \\ \exists y(\forall x(\varphi \to x \in y) \land \forall z(\forall x(\varphi \to x \in z) \to y \lessdot z)). \end{array} \right.$$

In words, $(\Diamond)$ asserts the existence for every formula $\varphi(x)$ of a *least* set containing '$\{x \mid \varphi(x)\}$'. We denote this (unique) set by $\{x \mid_\Diamond \varphi\}$.

If $\{x \mid_\Diamond \varphi\}$ = '$\{x \mid \varphi(x)\}$', we shall say that φ is *continuous* w.r.t. x. Note that we will use that terminology in $(\Diamond)$-free context as well, so that φ is continuous w.r.t. x simply means that '$\{x \mid \varphi(x)\}$' is a set. A central question is then to know which formulas can be continuous, and in a type-free setting this may be split up into two parts, not independent of each other: which *atomic formulas* can be continuous, and which *logical connectives* can preserve the continuity of a formula; these will be said to be *continuous* too.

Assuming $(\Diamond)$, $x = x$, $p = p$, and $\top$ are equally continuous w.r.t. x; they all define the universal set V. Notice also that $x \in p$ is clearly continuous

[1] This latter in particular is due to Olivier Esser.

w.r.t. x, seeing that $\{x \mid_\diamond x \in p\} = p$. The continuity of $p = x$ w.r.t. x is equivalent to the axiom $(\mathscr{A})$, and the one of $p \in x$ is $(\mathscr{B})$. These are going to be largely discussed. Finally, $x \in x$ may be continuous w.r.t. x only if $\neg$ is not continuous. And, of course, that $\neg$ preserves continuity is just $(\mathscr{C})$.

It is easy to see that $\wedge$ and $\forall$ are continuous under $(\diamond)$ once we notice that this latter is equivalent to the following scheme of *definable intersections*:

$$(\cap_\delta) : \quad \left| \begin{array}{l} \text{For every formula } \psi(z), \\ \exists y \forall x (x \in y \leftrightarrow \forall z(\psi \to x \in z)). \end{array} \right.$$

We denote this unique set y by $\bigcap`\{z \mid \psi(z)\}'$.

Proposition 4.1.1. $(\diamond) \Leftrightarrow (\cap_\delta)$.

Proof.
$\boxed{\Rightarrow}$: $\bigcap`\{z \mid \psi(z)\}' = \{x \mid_\diamond \forall z(\psi \to x \in z)\}$.
$\boxed{\Leftarrow}$: $\{x \mid_\diamond \varphi\} = \bigcap`\{z \mid \forall x(\varphi \to x \in z)\}'$. ⊣

Corollary 4.1.2. *Under $(\diamond)$, $\wedge$ and $\forall$ preserve the continuity of a formula.*

Proof. Assume $(\cap_\delta)$ and suppose that $\varphi(x, y)$ is continuous w.r.t. x. From $`\{x \mid \forall y \varphi(x, y)\}' = \bigcap`\{z \mid \exists y(z = `\{x \mid \varphi(x, y)\}')\}'$ it follows that $\forall y \varphi(x, y)$ is continuous w.r.t. $x$; and, obviously, that $\wedge$ preserves continuity results in the possibility of defining $p \cap q$ by $\bigcap`\{z \mid z = p \vee z = q\}'$. ⊣

It is worth stressing that Proposition 4.1.1. shows that $(\diamond)$ is just equivalent to particular instances of comprehension. Namely, $Comp[\varphi(x)]$ where $\varphi(x)$ is of the form '$\forall z(\psi \to x \in z)$', for any $\mathscr{L}$-formula ψ (with the proviso that x does not occur free in ψ).

By a *closure operator* on $D \subseteq \mathscr{P}(U)$ we mean a $\subseteq$-preserving application $(\cdot)^\diamond : D \longrightarrow D$ such that $A \subseteq A^\diamond$ and $(A^\diamond)^\diamond = A$, for any $A \in D$.[2] A subset A of U is said to be *closed* if it lies in $\mathrm{rng}(\cdot)^\diamond$, that is, if $A^\diamond = A$.

Given a structure $\mathcal{U}$, we recall that $\mathrm{def}[\cdot]_\mathcal{U}$ stands for the set of *definable* subsets. What follows is an elementary characterization of the models of $(\diamond)$.

Theorem 4.1.3. *A set-theoretic structure $\mathcal{U}$ is a model of $(\diamond)$ if and only if there exists a closure operator $(\cdot)^\diamond$ on $\mathrm{def}[\cdot]_\mathcal{U}$ such that $\mathrm{rng}[\cdot]_\mathcal{U} = \mathrm{rng}(\cdot)^\diamond$.*

Proof. $\langle x \mid \varphi(x, \bar{v})\rangle^\diamond_\mathcal{U} = [\{x \mid_\diamond \varphi\}^\mathcal{U}(\bar{v})]_\mathcal{U}$ for any formula $\varphi(x, \bar{p})$ and $\bar{v}$ in U. ⊣

[2] When $D = \mathscr{P}(U)$, we usually say a closure operator *on* U.

This simple observation will enable us to easily answer natural questions on some extensions of $(\Diamond)$ we are going to consider. Still more useful for constructing models is the second-order formulation of Theorem 4.1.3, in which $(\cdot)^\Diamond$ is now a closure operator on U and $(\Diamond)$ is replaced by $((\Diamond))$.[3]

In all the examples we consider, it is even the case that $(\cdot)^\Diamond$ is the closure operator associated with some (maybe trivial) *topology* on U, i.e. $(\cdot)^\Diamond$ further satisfies $\emptyset^\Diamond = \emptyset$ and $(A \cup B)^\Diamond = A^\Diamond \cup B^\Diamond$, for any $A, B \subseteq U$. This is, of course, what has motivated the title of this chapter. Notice that in such circumstances, Λ exists and $\vee$ is continuous too. It thus remains to explore $(\mathscr{A}), (\mathscr{B}), (\mathscr{C})$ and the continuity of $\exists$ under $(\Diamond)$.

Remark 4.1.1. The second-order version of Proposition 4.1.1 is the translation of the well-known correspondence between closure operators and (topped) intersection structures, which are complete lattices. Then, given a set-theoretic structure $\mathcal{U}$, what follows is immediate.

Fact 4.1.1. *If $\mathcal{U} \models ((\cap_\delta))$, then $\langle U; \lessdot_{\mathcal{U}} \rangle$ is a complete lattice.*

Proof. We observe that, for any $A \subseteq U$, $\bigwedge A := (\bigcap`\{z \mid z \in A\}')^{\mathcal{U}}$ is the infimum of A in $\langle U; \lessdot_{\mathcal{U}} \rangle$. $\dashv$

We would just notify the reader that the converse of this does not hold. The *spine model(s)* described in Chapter 5 will provide us with an illuminating example. All we are able to say when $\langle U; \lessdot_{\mathcal{U}} \rangle$ is a complete lattice is that, for any $A \subseteq U$, $\mathcal{U} \models \forall x(x \in \bigwedge A \to \forall z(z \in A \to z \in x))$.

4.2 Duality

The previous considerations naturally prompt us to examine the *symmetric* approach, according to which each class can be approximated from below. There is *a priori* no reason to prefer one view to the other. Besides, both views might equally well be taken simultaneously, which is going to be examined in the next sections.

The *dual* version of the approximation scheme $(\Diamond)$ is defined as follows

$$(\Box): \quad \left| \begin{array}{l} \text{For every formula } \varphi(x), \\ \exists y(\forall x(x \in y \to \varphi) \wedge \forall z(\forall x(x \in z \to \varphi) \to z \lessdot y)) \end{array} \right.$$

and the unique y given by $(\Box)$ is designated by $\{x \mid_\Box \varphi\}$.

[3] In the sequel, whenever we look at the second-order version of an axiom scheme, it will be designated with double parentheses '$((\,))$'.

The scheme of definable intersections is now to be replaced by the one of *definable unions*:

$$(\bigcup_\delta): \quad \left| \begin{array}{l} \text{For every formula } \psi(z), \\ \exists y \forall x (x \in y \leftrightarrow \exists z(\psi \wedge x \in z)) \end{array} \right.$$

and we denote this y by $\bigcup`\{z \mid \psi(z)\}'$.

Proposition 4.2.1. $(\Box) \Leftrightarrow (\bigcup_\delta)$.

Proof.
$\boxed{\Rightarrow}$: $\bigcup`\{z \mid \psi(z)\}' = \{x \mid_\Box \exists z(\psi \wedge x \in z)\}$.
$\boxed{\Leftarrow}$: $\{x \mid_\Box \varphi\} = \bigcup`\{z \mid \forall x(x \in z \rightarrow \varphi)\}'$. ⊣

Corollary 4.2.2. *Under* $(\Box)$, $\vee$ *and* $\exists$ *preserve the continuity of a formula.*

In the same way, one could give a characterization of the models of $(\Box)$ in terms of *interior* operators. We would leave this to the reader.

Naturally, from the semantic point of view, this duality just consists in taking the complement: we define the dual $\mathcal{U}^c$ of a set-theoretic structure $\mathcal{U}$ to be $\langle U\,; U \times U \setminus \in_{\mathcal{U}} \rangle$, so that $\mathcal{U} \models (\Diamond) \,/\, (\Box)$ if and only if $\mathcal{U}^c \models (\Box) \,/\, (\Diamond)$.

Assuming $(\mathscr{C})$, the duality is still more obvious on the axiomatic side.

Proposition 4.2.3. $(\Diamond) + (\mathscr{C}) \Leftrightarrow (\Box) + (\mathscr{C})$

Proof. $\{x \mid_\Box \varphi\} = \mathscr{C}(\{x \mid_\Diamond \neg\varphi\})$ and likewise by interchanging $\Diamond$ and $\Box$. ⊣

Surprisingly enough, if one is considering $(\mathscr{A})$, the symmetry is broken:

Fact 4.2.1. $(\Box) \Rightarrow$ *non*$(\mathscr{A})$, whereas $(\Diamond) + (\mathscr{A})$ is consistent.

Proof. Using $(\bigcup_\delta)$, we define $r := \bigcup`\{z \mid \exists w(z = \mathscr{A}(w) \wedge w \notin w)\}'$. Clearly, we have $r \in r \rightarrow r \notin r$, and if $\mathscr{A}(r)$ existed, we would too have $r \notin r \rightarrow r \in r$. For the consistency of $(\Diamond) + (\mathscr{A})$, we rely on the existence of N_ω. ⊣

Despite the loss of singletons, we are going to be interested in any possible set theory based upon both $(\Diamond)$ and $(\Box)$. This was initiated in [34], where $(\mathscr{C})$ is taken too, and then reinvestigated in [31] without assuming $(\mathscr{C})$ anymore.

4.3 The comprehension scheme revisited

By taking ($\Box$), we must give up ($\mathscr{A}$). Nevertheless, we may use ($\Diamond$) to define what we call *pseudo-singletons* or *molecules* (as named in [31]). By definition, the pseudo-singleton of p is the smallest set containing the singleton of p, that is, $\{x \mid_\diamond p = x\} = \bigcap \mathscr{B}(p)$, and this is nothing but $\mathscr{M}(p)$.

Proposition 4.3.1. $(\Diamond) + (\Box) \Leftrightarrow (\mathscr{M}) + (\Box)$.

Proof.
$\boxed{\Rightarrow}$: Obvious.
$\boxed{\Leftarrow}$: For any formula $\varphi(x)$, $\{x \mid_\diamond \varphi\} = \bigcup`\{z \mid \exists w(z = \mathscr{M}(w) \wedge \varphi(w))\}'$. ⊣

The right to left part of the proof is interesting in that it shows that $\{x \mid_\diamond \varphi\} = `\{x \mid \exists w(w \mathrel{\dot{\lessgtr}} x \wedge \varphi(w))\}'$, which leads to defining the scheme

$$(\triangle): \quad \left|\begin{array}{l} \text{For every formula } \varphi(x), \\ \exists y \forall x (x \in y \leftrightarrow \exists w (w \mathrel{\dot{\lessgtr}} x \wedge \varphi(w))) \end{array}\right.$$

and then proving that

Proposition 4.3.2. $(\triangle) \Leftrightarrow (\Diamond) + (\Box)$.

Proof.
$\boxed{\Leftarrow}$: $y = \{x \mid_\diamond \varphi\}$
$\boxed{\Rightarrow}$: If $\varphi(x)$ is taken to be $\exists z(x \in z \wedge \psi(z))$ in ($\triangle$), what we get is just ($\cup_\delta$). And if $\varphi(x)$ is taken to be $x = p$, we have ($\mathscr{M}$). ⊣

The axiom scheme ($\triangle$) should not leave the reader indifferent, for if $\dot{\lessgtr}$ was the equality - which is the case when ($\mathscr{A}$) holds (cf. Fact 2.1.1) - it would be nothing but the *full* comprehension scheme. But that one would be willing to sacrifice ($\mathscr{A}$) just because ($\triangle$) is provably consistent would hardly be arguable in view of what follows.

Given a preorder R on a set U, we denote by $(\cdot)^R$ the *closure* operator on U that is defined by $A^R := R``A$; and then, for any $D \subseteq \mathscr{P}(U)$, we let $D^\uparrow$ stand for $\{A^R \mid A \in D\}$ and $D^\downarrow$ for $\{A^{R^{-1}} \mid A \in D\}$.

Given a set-theoretic structure $\mathcal{U}$, $\dot{\lessgtr}_\mathcal{U}$ is a preorder on U, and as it is definable, we have $\mathrm{def}[\cdot]^\uparrow_\mathcal{U} \subseteq \mathrm{def}[\cdot]_\mathcal{U}$ and $\mathrm{def}[\cdot]^\downarrow_\mathcal{U} \subseteq \mathrm{def}[\cdot]_\mathcal{U}$, so that $(\cdot)^{\dot{\lessgtr}_\mathcal{U}}$ is in particular a closure operator on $\mathrm{def}[\cdot]_\mathcal{U}$.

We are ready to formulate the semantic counterpart of Proposition 4.3.2.

Theorem 4.3.3. *A model $\mathcal{U}$ of ($\Diamond$) is a model of ($\Box$) if and only if the closure operator $(\cdot)^\Diamond$ given by Theorem 4.1.3 coincides with $(\cdot)^{\dot{\lessgtr}_\mathcal{U}}$, and thus* $\mathrm{rng}[\cdot]_\mathcal{U} = \mathrm{def}[\cdot]^\uparrow_\mathcal{U}$.

Proof. Assuming $(\Diamond)$, we have seen that $\{x \mid_\diamond \varphi\} =$ '$\{x \mid \exists w(w \mathrel{\dot\leqslant} x \wedge \varphi(w))\}$' if and only if $(\Box)$ holds. ⊣

Note that in any case, if $(\cdot)^\Diamond$ was of the form $(\cdot)^R$ for some preorder R on U, this should be $\dot\leqslant_{\mathcal{U}}$:

Proof. Suppose $u \mathrel{\dot\leqslant_{\mathcal{U}}} v$. As $\{u\}^R$ is closed, there exists $w \in U$ such that $[w]_{\mathcal{U}} = \{u\}^R$, and then, as $u \in \{u\}^R$, that is, $u \in_{\mathcal{U}} w$, we get $v \in_{\mathcal{U}} w$, that is, $v \in \{u\}^R$. Whence $u\ R\ v$. Conversely, suppose $u\ R\ v$ and let $w \in U$ such that $u \in_{\mathcal{U}} w$, that is, $u \in [w]_{\mathcal{U}}$. Now, as $u\ R\ v$ and $[w]_{\mathcal{U}}$ is closed, i.e. $R``[w]_{\mathcal{U}} = [w]_{\mathcal{U}}$, we have $v \in [w]_{\mathcal{U}}$ too, that is, $v \in_{\mathcal{U}} w$. Therefore $u \mathrel{\dot\leqslant_{\mathcal{U}}} v$. ⊣

We thus may state the second-order version of Theorem 4.3.3 as follows

Theorem 4.3.4. *A set-theoretic structure $\mathcal{U}$ is a model of $(\!(\triangle)\!)$ if and only if there exists a preorder R on U such that* $\operatorname{rng}[\cdot]_{\mathcal{U}} = \mathscr{P}(U)^{\uparrow}$. *In that case, R must coincide with $\dot\leqslant_{\mathcal{U}}$.*

With this, it is rather child's play to concoct models of $(\!(\triangle)\!)$, and in particular finite ones, which would testify of the insignificance of $(\triangle)$ as a set theory on its own. Incidentally, there was no need to invoke Theorem 4.3.4 to prove its consistency, as '$\{\Lambda, \mathrm{V}\}$' is obviously the simplest model, and this is also a model for the situation we examine hereafter.

4.4 The symmetric case

Without further assumptions, all the properties that $\dot\leqslant$ is certain to possess are *reflexivity* and *transitivity*. According to Fact 2.1.1, when $(\mathscr{A})$ fails, $\dot\leqslant$ may be thought of as a reminiscence of the equality. Then it would be legitimate to require the *symmetry* of $\dot\leqslant$. Under $(\triangle)$, this actually amounts to demanding that the negation preserves the continuity of a formula.

Proposition 4.4.1.

i) $(\mathscr{C}) \Rightarrow (x \mathrel{\dot\leqslant} y \to y \mathrel{\dot\leqslant} x)$.

ii) Assuming $(\triangle)$, $(x \mathrel{\dot\leqslant} y \to y \mathrel{\dot\leqslant} x) \Rightarrow (\mathscr{C})$.

Proof.
i) Suppose $x \mathrel{\dot\leqslant} y$. If $x \notin z$, then $x \in \mathscr{C}(z)$, and so $y \in \mathscr{C}(z)$, that is, $y \notin z$.
ii) Suppose that $(x \mathrel{\dot\leqslant} y \to y \mathrel{\dot\leqslant} x)$. We show that $\{x \mid_\diamond x \notin p\} = \mathscr{C}(p)$.
Let $x \in \{x \mid_\diamond x \notin p\}$. By $(\triangle)$, there exists w such that $w \mathrel{\dot\leqslant} x$ and $w \notin p$. Then $x \mathrel{\dot\leqslant} w$ and it follows that $x \notin p$ either, that is, $x \in \mathscr{C}(p)$. ⊣

To sum up, a model $\mathcal{U}$ of $(\triangle)$ satisfies $(\mathscr{C})$ if and only if $\dot{\leqslant}_{\mathcal{U}}$ is an equivalence relation, namely $\doteq_{\mathcal{U}}$. Thence we can give the next useful characterization which falls out of Theorem 4.3.4.

Theorem 4.4.2. *A set-theoretic structure $\mathcal{U}$ is a model of $((\triangle)) + (\mathscr{C})$ if and only if there exists an equivalence relation R on U together with a bijection $f : U \longrightarrow \mathscr{P}(U/_R)$ such that $[u]_{\mathcal{U}} = \bigcup f`u$ for each $u \in U$. In that case, R must coincide with $\doteq_{\mathcal{U}}$.*

Proof. Observe that $(\cdot)^R$-closed subsets are just unions of equivalences classes, and thus we have to have $f`u := \{\{v\}^R \mid v \in [u]_{\mathcal{U}}\}$, for any $u \in U$. ⊣

The simplified version that follows speaks for itself:

Theorem 4.4.3. *A set U is the universe of a model of $((\triangle)) + (\mathscr{C})$ if and only if $|U| = 2^\kappa$, for some cardinal κ.*

Proof.
Necessity: Follows directly from Theorem 4.4.2.
Sufficiency: Take any equivalence relation R on U such that $|U/_R| = \kappa$, and then any bijection $f : U \longrightarrow \mathscr{P}(U/_R)$ to define $[\cdot]_{\mathcal{U}}$ as in Theorem 4.4.2. ⊣

At least, in assuming $(\triangle)+(\mathscr{C})$, all the basic logical connectives $(\neg, \wedge, \vee, \forall, \exists)$ are continuous, but this results in serious drawbacks at the atomic level. We already know that $(\mathscr{A})$ is incompatible with $(\triangle)$. It turns out that $(\mathscr{B})$ is also incompatible with $(\triangle) + (\mathscr{C})$, and this can happen in a $(\triangle)$-free context:

Fact 4.4.1. *If $\neg$, $\wedge$, $\exists$ are continuous, then $(\mathscr{B})$ is inconsistent.*

Proof. Under the assumptions, $x \dot{\leqslant} y \equiv \neg\exists z(x \in z \wedge y \notin z)$ is continuous w.r.t. x, and then, if we assume that $y \in x$ is continuous w.r.t. x, so is $\exists y(x \dot{\leqslant} y \wedge y \notin x)$. Now, let r be the set defined by this formula. It is easy to see that $r \in r$ if and only if $r \notin r$. ⊣

Note that if the use of quantifiers is forbidden, this is no longer provable:

Fact 4.4.2. *$(\mathscr{B})$ is compatible with the use of $\neg$ and $\wedge$ in formulas defining sets.*

A proof of this can be found in Skolem's paper [38] (Theorem 2, p. 165).

Thus $(\mathscr{A})$, $(\mathscr{B})$, and obviously (W) are all incompatible with $(\triangle) + (\mathscr{C})$. Note incidentally that we are going to show in the next section that, as for $(\mathscr{A})$, $(\mathscr{B})$ is already incompatible with $(\triangle)$ alone (this is not immediate).
To conclude and summarize, the next result - which is not mentioned in [34] or [31] - clearly states what the continuous formulas are under $(\triangle)+(\mathscr{C})$.

Theorem 4.4.4. $(\triangle) + (\mathscr{C}) \Leftrightarrow$ *Comp*$[\varphi(x)]$ *for any formula* $\varphi(x)$ *in* $\mathscr{L}_*$ *with the sole restriction that the abstracted variable* x *does not occur on the right-hand side of* $\in$.

Proof.
$\boxed{\Rightarrow}$: Follows from the continuity of all the logical connectives under $(\triangle) + (\mathscr{C})$, and the fact that the limitation to $\mathscr{L}_*$-formulas and the restriction on x prevent occurrences of $p = x$, $p \in x$, $x \in x$ at the atomic level.
$\boxed{\Leftarrow}$: Observe that $\bigcap$'$\{z \mid \psi(z)\}$' = '$\{x \mid \forall z(\psi \to x \in z\}$' is just defined by such a suitable formula (note that as we assume extensionality, ψ can be reduced to an $\mathscr{L}_*$-formula by replacing any occurrence of $=$ by $\underset{\cdot}{=}$). Thus we get $(\bigcap_\delta)$. And that we have $(\mathscr{C})$ is still more obvious. ⊣

It should be stressed that, as well as the absence of $=$, this simple restriction on the abstracted variable has disastrous effects on the development of set theory, as for instance it prevents us from defining the power-set of any given set p, seeing that $\mathscr{P}(p) =$ '$\{x \mid \forall z(z \in x \to z \in p)\}$'.

4.5 The antisymmetric case

The absence of $(\mathscr{C})$ seems to leave the door open to the continuity of $x \in x$ and $p \in x$ w.r.t. x. In fact, that (W) is compatible with $(\triangle)$ is easily seen:

Example 4.5.1. Take $U := \{a, b, c\}$ with $R := U \times U \setminus \{(b, a), (c, a)\}$, i.e. $\{a\}^R = \{a, b, c\}$, $\{b\}^R = \{b, c\} = \{c\}^R$ (so R is neither symmetric nor antisymmetric), and define $[\cdot]_{\mathcal{U}}$ as follows: $[a]_{\mathcal{U}} := \emptyset$, $[b]_{\mathcal{U}} := \{b, c\}$, $[c]_{\mathcal{U}} := U$. Thus, $\mathcal{U}$ is a model of $(\!(\triangle)\!)$ in which a is Λ, b is W, and c is V. Notice that both $\mathscr{B}(b)$ and $\mathscr{B}(c)$ exist and are equal to b, but $\mathscr{B}(a)$ does not exist in $\mathcal{U}$.

Trying to find a model of $(\mathscr{B})$ is directly a less obvious task for there is no finite model of $(\mathscr{B})$ (see Chapter 5). Under $(\triangle)$, the task will soon prove vain. To proceed, we first point out the following result (to be compared with Proposition 4.4.1).

Proposition 4.5.1.

i) $(\mathscr{B}) \Rightarrow (x \dot{\leqslant} y \to x \underset{\cdot}{\leqslant} y)$.

ii) Assuming $(\triangle)$, $(x \dot{\leqslant} y \to x \underset{\cdot}{\leqslant} y) \Rightarrow (\mathscr{B})$.

Proof.
i) Suppose $x \dot{\leqslant} y$. If $z \in x$, then $x \in \mathscr{B}(z)$, and so $y \in \mathscr{B}(z)$, that is, $z \in y$.
ii) Suppose that $(x \dot{\leqslant} y \to y \underset{\cdot}{\leqslant} x)$. We show that $\{x \mid_\diamond p \in x\} = \mathscr{B}(p)$.
Let $x \in \{x \mid_\diamond p \in x\}$. By $(\triangle)$, there exists w such that $w \dot{\leqslant} x$ and $p \in w$. Then $w \underset{\cdot}{\leqslant} x$ and it follows that $p \in x$ too, that is, $x \in \mathscr{B}(p)$. ⊣

We are going to show that $(x \mathrel{\dot{\leqslant}} y \to x \mathrel{\underset{\cdot}{\leqslant}} y)$ is incompatible with $(\triangle)$. In order to achieve this, we need a couple of lemmas.

Given a set-theoretic structure $\mathcal{U}$, we say that a function $f : U \longrightarrow \mathscr{P}(U)$ is *definable* if there exists an $\mathscr{L}$-formula $\varphi(x, y, \bar{p})$ and $\bar{v} \in U$ such that for all $u \in U$, $f\text{‘}u = \langle x \mid \varphi(x, u, \bar{v})\rangle_{\mathcal{U}}$.

Lemma 4.5.2. *Let $\mathcal{U}$ be a set-theoretic structure. If there exists a surjective definable function $f : U \longrightarrow \mathrm{def}[\cdot]^{\uparrow}_{\mathcal{U}}$ such that $u \mathrel{\dot{\leqslant}_{\mathcal{U}}} v \Rightarrow f\text{‘}u \subseteq f\text{‘}v$, then there exists such a surjective definable function $g : U \longrightarrow \mathrm{def}[\cdot]^{\downarrow}_{\mathcal{U}}$*

Proof. We define $g\text{‘}u$ for any $u \in U$ by $U \setminus \bigcup\{f\text{‘}w \mid w \notin f\text{‘}u\}$, that is, for all x in U, $x \in g\text{‘}u \Leftrightarrow \forall w \in U\,(x \in f\text{‘}w \to w \in f\text{‘}u)$. Therefrom it is easy to see that $g : U \longrightarrow \mathrm{def}[\cdot]^{\downarrow}_{\mathcal{U}}$ is definable and satisfies $u \mathrel{\dot{\leqslant}_{\mathcal{U}}} v \Rightarrow g\text{‘}u \subseteq g\text{‘}v$. It remains to show that g is surjective. Take $A \in \mathrm{def}[\cdot]^{\downarrow}_{\mathcal{U}}$. Clearly, $U \setminus A \in \mathrm{def}[\cdot]^{\uparrow}_{\mathcal{U}}$ and so, as f is surjective, we can choose $b \in U$ such that $f\text{‘}b = U \setminus A$. Set $B := \{b\}^{\dot{\geqslant}_{\mathcal{U}}}$. As $B \in \mathrm{def}[\cdot]^{\downarrow}_{\mathcal{U}}$, we have $U \setminus B \in \mathrm{def}[\cdot]^{\uparrow}_{\mathcal{U}}$, and so, again, we can choose $a \in U$ such that $f\text{‘}a = U \setminus B$. Now, we do have $g\text{‘}a = A$, for $\bigcup\{f\text{‘}w \mid w \notin f\text{‘}a\} = \bigcup\{f\text{‘}w \mid w \in B\} = \bigcup\{f\text{‘}w \mid w \mathrel{\dot{\leqslant}_{\mathcal{U}}} b\} = f\text{‘}b = U \setminus A$. ⊣

Lemma 4.5.3. *Given a set-theoretic structure $\mathcal{U}$, there is no surjective definable function $g : U \longrightarrow \mathrm{def}[\cdot]^{\downarrow}_{\mathcal{U}}$ such that $u \mathrel{\dot{\leqslant}_{\mathcal{U}}} v \Rightarrow g\text{‘}u \subseteq g\text{‘}v$.*

Proof. Let g be as above. Define $R = \{u \in U \mid u \notin g\text{‘}u\}^{\dot{\geqslant}_{\mathcal{U}}}$. Clearly, $R \in \mathrm{def}[\cdot]^{\downarrow}_{\mathcal{U}}$ and so there exists $r \in U$ such that $g\text{‘}r = R$. If we had $r \notin R$, we would too have $r \in R$ by definition of R. Whence $r \in R$. But then there exists $u_0 \in U$ such that $u_0 \notin g\text{‘}u_0$ and $r \mathrel{\dot{\leqslant}_{\mathcal{U}}} u_0$. From this latter we get $g\text{‘}r \subseteq g\text{‘}u_0$, and thus, as $u_0 \in R = g\text{‘}r$, we have $u_0 \in g\text{‘}u_0$ as well. Contradiction. ⊣

Theorem 4.5.4. $(x \mathrel{\dot{\leqslant}} y \to x \mathrel{\underset{\cdot}{\leqslant}} y)$ *- and so* $(\mathscr{B})$ *- is incompatible with* $(\triangle)$

Proof. For suppose there exists $\mathcal{U}$ which is a model of $(\triangle)+(x \mathrel{\dot{\leqslant}} y \Rightarrow x \mathrel{\underset{\cdot}{\leqslant}} y)$. Then $[\cdot]_{\mathcal{U}}$ would be a surjective (definable) function $U \longrightarrow \mathrm{def}[\cdot]^{\uparrow}_{\mathcal{U}}$ such that $u \mathrel{\dot{\leqslant}_{\mathcal{U}}} v \Rightarrow [u]_{\mathcal{U}} \subseteq [v]_{\mathcal{U}}$, which is impossible in view of Lemmas 4.5.2–3. ⊣

It does not seem possible to find a *syntactical* proof of Theorem 4.5.4. Such a proof, however, can be given for a slight variant of it.

Proposition 4.5.5. $(x \mathrel{\dot{\leqslant}} y \to y \mathrel{\underset{\cdot}{\leqslant}} x)$ *is incompatible with* $(\triangle)$

Proof. Let r be $\{x \mid_{\diamond} x \notin x\}$, which, by $(\triangle)$, is '$\{x \mid \exists w(w \mathrel{\dot{\leqslant}} x \wedge w \notin w)\}$'. Clearly $r \notin r \to r \in r$, so we must have $r \in r$. Then there exists w such that $w \mathrel{\dot{\leqslant}} r$ and $w \notin w$. If we assumed $(x \mathrel{\dot{\leqslant}} y \Rightarrow y \mathrel{\underset{\cdot}{\leqslant}} x)$, it would result that $w \notin r$, which is impossible for $w \in r$ (since $w \notin w$). ⊣

We mention that the converse of $(x \mathrel{\dot{\leqslant}} y \rightarrow x \leqslant y)$ is obviously consistent with $(\triangle)$, as it holds in the two-points models '$\{\dot{\Lambda}, \mathrm{V}\}$' or even in the model given in Example 4.5.1. But this is a rather odd principle, as, for instance, it is apparent that '$\{\Lambda\}$' cannot exist under $(x \mathrel{\underset{\cdot}{\leqslant}} y \rightarrow x \mathrel{\dot{\leqslant}} y)$, which would not really speak for any set theory in which this latter holds. Nevertheless, here is an example about which we should not be indifferent.

Proposition 4.5.6. *$Abst[\mathscr{L}^{+}_{\tau *}] + Ext \Rightarrow (x \mathrel{\underset{\cdot}{\leqslant}} y \rightarrow x \mathrel{\dot{\leqslant}} y)$.*

Proof. Assume $Abst[\mathscr{L}^{+}_{\tau *}]$ and let p, q be such that $p \mathrel{\underset{\cdot}{\leqslant}} q$ but $p \mathrel{\dot{\nleqslant}} q$. Define $\sigma(x) := \{z \mid z \in p \vee (x \in x \wedge z \in q)\}$. It easily follows from $p \mathrel{\underset{\cdot}{\leqslant}} q$ that $\forall x((x \in x \rightarrow \sigma(x) = q) \wedge (x \notin x \rightarrow \sigma(x) = p))$ (Ext is needed for this). Now, as $p \mathrel{\dot{\nleqslant}} q$, let us choose r with $p \in r, q \notin r$, and define $\rho = \{x \mid \sigma(x) \in r\}$. Thus we have $\forall x(x \in \rho \leftrightarrow x \notin x)$, and therefore $\rho \in \rho \leftrightarrow \rho \notin \rho$. ⊣

Although we have assumed Ext throughout this chapter, it was important to mention its use in Proposition 4.5.6, for otherwise one might have been tempted to think that $Abst[\mathscr{L}^{+}_{\tau *}] \Rightarrow (x \mathrel{\underset{\cdot}{=}} y \rightarrow x \doteq y)$, which is false. As we shall see in Chapter 6, there are *term models* for $Abst[\mathscr{L}^{+}_{\tau *}]$ in which $^{\flat}Ext$ fails, as well as there are in which it holds, and the proof of this is particularly not obvious. The consistency of $Abst[\mathscr{L}^{+}_{\tau *}] + Ext$ will fall out more easily of Chapter 5 in which we show that $Abst[\mathscr{L}^{+}_{\tau *}]$ - and so $(\mathscr{B})$ - is at least compatible with $(\diamond)$ and $(\square)$ separately.

In assuming $(\mathscr{B})$ (and Ext), $\dot{\leqslant}$ is *anti-symmetric*, i.e. $(x \doteq y \rightarrow x = y)$. It is then legitimate to enquire whether this natural principle could not be compatible with $(\triangle)$. At least, it would prevent the existence of finite models.

Fact 4.5.1. Any model $\mathcal{U}$ of $(\Lambda) + (\mathscr{M}) + (x \doteq y \rightarrow x = y)$ is infinite.

Proof. Notice that $x \doteq y \Leftrightarrow \mathscr{M}(x) = \mathscr{M}(y)$ and that $\Lambda = \mathscr{M}(x)$ for no x. Then it is clear that, under the assumptions, we can define a potentially infinite sequence of elements in U by iterating $\mathscr{M}(\cdot)$ from $\Lambda^{\mathcal{U}}$. ⊣

The consistency of $(\triangle) + (x \doteq y \rightarrow x = y)$ is raised and left open in [31]. By invoking Theorem 4.3.4, it is now fairly easy to give a positive answer.

Example 4.5.2. Take $U := \mathbb{N}$, R the usual ordering $\leqslant$ on $\mathbb{N}$, and define $[\cdot]_{\mathcal{U}}$ as follows: $[0]_{\mathcal{U}} := \emptyset$, $[n]_{\mathcal{U}} := \{m \mid n - 1 \leqslant m\}$, for any $n \geqslant 1$. Thus it is clear that $\mathrm{rng}[\cdot]_{\mathcal{U}}$ is just the collection of $(\cdot)^{R}$-closed subsets of U, so $\mathcal{U} \models ((\triangle))$. In $\mathcal{U}$, 0 is Λ, 1 is V, and 2 is W, which shows that this latter can exist when $\dot{\leqslant}$ is antisymmetric. On the other hand, note that $\mathscr{B}(u)$ exists for no u in $\mathcal{U}$.

4.6 Normality

Assuming ($\Box$), not every singleton can exist, so we shall say that a set x is *normal* if '$\{x\}$' exists. No surprises, as we show in this section, very few things can be said about the class of normal sets, which is defined by

$$\mathscr{N} := `\{x \mid \exists y(y = \mathscr{A}(x))\}' .$$

Proposition 4.6.1. *$\exists x(x \notin \mathscr{N})$.*

Proof. Just a reformulation of the first part of Fact 4.2.1. ⊣

On the other hand, $\exists x(x \in \mathscr{N})$ is not even derivable from ($\triangle$) + ($\mathscr{C}$), for in the two-points model '$\{\Lambda, \mathrm{V}\}$' we have $\mathscr{N} = \Lambda$. As certified by this example, it can be shown, at least, that $\mathscr{N}$ is always a set.

Lemma 4.6.2. *For any formula $\varphi(x)$, if $z \in \mathscr{N}$ and $\varphi(z)$, then $z \in \{x \mid_{\Box} \varphi\}$.*

Proof. Suppose $\varphi(z)$ and $z \in \mathscr{N}$. Then, $\{x \mid_{\Box} \varphi\} \cup \mathscr{A}(z)$ is a set, and as we have $\forall x(x \in \{x \mid_{\Box} \varphi\} \cup \mathscr{A}(z) \rightarrow \varphi)$, it follows that $\{x \mid_{\Box} \varphi\} \cup \mathscr{A}(z) \lessdot \{x \mid_{\Box} \varphi\}$. Whence $z \in \{x \mid_{\Box} \varphi\}$. ⊣

Proposition 4.6.3. *$\mathscr{N}$ is a set.*

Proof. Take $\varphi(x)$ to be $\exists y(y = \mathscr{A}(x))$ in the previous lemma. ⊣

Such a simple question as to know whether $\mathscr{N} \in \mathscr{N}$ or not is undecidable. In the two-points model, we have $\mathscr{N} = \Lambda$, so that $\mathscr{N} \notin \mathscr{N}$ is consistent; and by invoking Theorem 4.4.2, it is easy to concoct a model $\mathcal{U}$ of ($\triangle$) + ($\mathscr{C}$) in which, for instance, $\mathscr{N} = \mathscr{A}(\mathscr{N})$, so that $\mathscr{N} \in \mathscr{N}$ is also consistent.

Example 4.6.1. Take $U = \{a, b, c, d\}$, with $\{d\}$ and $\{a, b, c\}$ as R-classes, and with $[a]_{\mathcal{U}} := \emptyset, [b]_{\mathcal{U}} := \{a, b, c\}, [c]_{\mathcal{U}} := U, [d]_{\mathcal{U}} := \{d\}$. In $\mathcal{U}$, d is $\mathscr{N}$.

With the help of Lemma 4.6.2, we can also give an eloquent characterization of abnormal sets. These are just *tokens* of the discontinuity of a formula, in the following sense:

Proposition 4.6.4. *$z \notin \mathscr{N}$ if and only if there exists a formula $\varphi(x)$ such that $\varphi(z)$ but $z \notin \{x \mid_{\Box} \varphi\}$.*

Proof.
Necessity: Just take $\varphi(x)$ to be $z = x$.
Sufficiency: Use Lemma 4.6.2. ⊣

Perhaps the most significant manifestation of the absence of control over $\mathscr{N}$ is the following seemingly positive result, which states that $\mathscr{N}$ may be taken to be any preexisting universe of normal sets (e.g. a model of ZF).

Theorem 4.6.5. *Let $\mathcal{V}$ be any infinite extensional set-theoretic structure satisfying $(\mathscr{A})$. Then there exists a model $\mathcal{U}$ of $((\triangle)) + (\mathscr{C})$ such that:*

(i) $[\cdot]_{\mathcal{V}} = [\cdot]_{\mathcal{U}}$ *restricted on V ;*

(ii) $V = \{u \in U \mid \mathcal{U} \models u \in \mathscr{N}\}$ *;*

(iii) $\mathscr{P}(V) \subseteq \mathrm{rng}[\cdot]_{\mathcal{U}}$.

Proof. Let U be $V \cup V' \cup V''$, where $V' := \mathscr{P}(V) \setminus \mathrm{rng}[\cdot]_{\mathcal{V}}$ and V'' is any set of cardinality $2^{|V|}$, and where we also assume that V, V', V'' are pairwise disjoint. We now equip each of these with an equivalence relation: we define S on V by $V/_S := \{\{v\} \mid v \in V\}$, S' on V' by $V'/_{S'} := \{V'\}$, and we take any equivalence S'' on V'' such that $|V''/_{S''}| = |V|$ and $|K| \geqslant 2$, for all $K \in V''/_{S''}$. Then we let R stand for the equivalence on U defined by $S \cup S' \cup S''$. Notice that $|U| = |\mathscr{P}(U/_R)| = 2^{|V|}$. Thus, if we set $f`v := \{\{w\} \mid w \in_{\mathcal{V}} v\}$ for each $v \in V$ and $f`W := \{\{w\} \mid w \in W\}$ for all $W \in V'$, then f can be so extended over U as to define a bijection $U \longrightarrow \mathscr{P}(U/_R)$. By invoking Theorem 4.4.2, we can now turn U into a set-theoretic structure $\mathcal{U} \models ((\triangle)) + (\mathscr{C})$ by setting $[u]_{\mathcal{U}} := \bigcup f`u$ for any $u \in U$. It remains to check that $\mathcal{U}$ satisfies $(i), (ii), (iii)$. For any $v \in V$, we have $[v]_{\mathcal{U}} = \bigcup\{\{w\} \mid w \in_{\mathcal{V}} v\} = \{w \mid w \in_{\mathcal{V}} v\}$, and this is just $[v]_{\mathcal{V}}$. In particular, since $\mathcal{V} \models (\mathscr{A})$, we have $\{v\} = [\mathscr{A}^{\mathcal{V}}(v)]_{\mathcal{V}} = [\mathscr{A}^{\mathcal{V}}(v)]_{\mathcal{U}}$, which shows that $\mathcal{U} \models v \in \mathscr{N}$ and that $\mathscr{A}^{\mathcal{U}}(v) = \mathscr{A}^{\mathcal{V}}(v)$, for any $v \in V$. Conversely, suppose $\mathcal{U} \models u \in \mathscr{N}$, that is, $\{u\} = \bigcup A$ for some $A \subseteq U/_R$. If we had $A \cap (V'/_{S'} \cup V''/_{S''}) \neq \emptyset$, we would have $|\bigcup A| \geqslant 2$, for V' is infinite and $|K| \geqslant 2$ for any $K \in V''/_{S''}$. Therefore we must have $A \subseteq V/_S$, and so $\{u\} \subseteq V$. Finally, let $W \subseteq \mathscr{P}(V)$. If $W \in \mathrm{rng}[\cdot]_{\mathcal{V}}$, then clearly $W \in \mathrm{rng}[\cdot]_{\mathcal{U}}$. If $W \notin \mathrm{rng}[\cdot]_{\mathcal{V}}$, $W \in V'$, and then $[W]_{\mathcal{U}} = \bigcup f`W = W$, so $W \in \mathrm{rng}[\cdot]_{\mathcal{U}}$. ⊣

Has the scheme $(\triangle) + (\mathscr{C})$ been to some extent salvaged by this result? In view of the arbitrary nature of the model constructed in the proof, it seems not. A possible way to try to define a notion of *coherence* in models of $(\triangle) + (\mathscr{C})$ is discussed in the next and last section, in which we summarize the mathematical content of this chapter.

4.7 Coherence

Given a topological space U, we recall that $\mathscr{P}_{cl}(U)$ stand for the set of *closed* subsets of U, and $\mathscr{P}_{op}(U)$ for the set of *open* ones.

Clearly, according to the second-order version of Theorem 4.1.3 / its dual, any topological space U such that $U \simeq \mathscr{P}_{cl}(U)$ / $U \simeq \mathscr{P}_{op}(U)$ gives rise to a model of $((\Diamond))/((\Box))$. Notice that the closure/interior operator attached to a given model of $((\Diamond))/((\Box))$ need not be topological. For our purposes, however, we shall restrict ourselves to topological models. But even in that case there are still many insignificant models of $((\Diamond))/((\Box))$, in particular finite ones. So in what remains of this chapter we shall rather be concerned with topological models that are solutions to $U \cong \mathscr{P}_{cl}(U)$ / $U \cong \mathscr{P}_{op}(U)$ within specific categories of topological spaces. Anyhow, as we aim to show, the existence of such solutions is closely related to the consistency problem of some natural extensions of $((\Diamond))/((\Box))$. It may also be said that such solutions $\mathcal{U}$ to $U \cong \mathscr{P}_{cl}(U)$ / $U \cong \mathscr{P}_{op}(U)$ define *natural* models for $((\Diamond))/((\Box))$, in that the extension function $[\cdot]_{\mathcal{U}}$ is to be not only a bijection but a *homeomorphism*, which does guarantee that there is some *coherence* in the process of assigning extensions to sets. It is understood here that $\mathscr{P}_{cl}(U)$ / $\mathscr{P}_{op}(U)$ has itself been equipped with a *natural* topological structure derived from the one of U.

Alexandroff spaces

Any model of $((\triangle))$ is topological. This is a direct consequence of Theorem 4.3.4 which states that such a model appears as a preordered set $\langle U; R\rangle$ such that $U \simeq \mathscr{P}(U)^{\uparrow}$; and this latter is just $\mathscr{P}_{op}(U)$ when U is endowed with the *Alexandroff* topology. Notice that, as $\mathscr{P}(U)^{\uparrow} = \mathscr{P}(U^*)^{\downarrow}$ where U^* is $\langle U; R^{-1}\rangle$, any model of $((\triangle))$ may equally be viewed as a topological solution to $U \simeq \mathscr{P}_{cl}(U)$, where U is now endowed with the Alexandroff topology of U^*. This is what is implicit in Theorem 4.3.4, for $(\cdot)^R$ is a closure operator, not an interior one. Still, it is more natural to view $\mathscr{P}(U)^{\uparrow}$ as $\mathscr{P}_{op}(U)$. The reason why is that then R - and so $\dot{\leqslant}_{\mathcal{U}}$ - coincides with the so-called *specialization preorder* $\lhd_U$ of the topology, which is defined by $u \lhd_U v \Leftrightarrow \forall A \in \mathscr{P}_{op}(U)(u \in A \rightarrow v \in A)$. This is also referred to as the *indiscernibility relation* associated with the topology, for $u \ntriangleleft_U v \Leftrightarrow \exists A \in \mathscr{P}_{op}(U)(u \in A \wedge v \notin A) \Leftrightarrow$ 'u is *discernible* from v', in topological terms.

It is clear that for any topological space U we have $\mathscr{P}_{op}(U) \subseteq \mathscr{P}(U)^{\uparrow}$, where this latter is taken with respect to $\lhd_U$; and then it is easy to see that a topological space U will satisfy $\mathscr{P}_{op}(U) = \mathscr{P}(U)^{\uparrow}$ if and only if $\mathscr{P}_{op}(U)$ is closed under taking arbitrary intersections, that is to say, if this latter also defines the set of closed subsets for some topology on U. We call those topological spaces *Alexandroff spaces*. Their topology is thus generated by a preorder, which must coincides with the specialization preorder.

All that to say that a model of $((\triangle))$ is just a solution $\mathcal{U}$ to $U \simeq \mathscr{P}_{op}(U)$ within $ALEX$, the category of Alexandroff spaces. The Alexandroff-continuous functions are the functions that preserve the indiscernibility relation.

Given an Alexandroff space U, a possible way to transfer the indiscernibility relation onto $\mathscr{P}(U)$, and thus to turn this latter into an Alexandroff space, is by defining $A \lhd_{\mathscr{P}(U)} B$ if and only if $A \subseteq B^{\lhd_U}$. The restriction of this to $\mathscr{P}_{op}(U) = \mathscr{P}(U)^{\uparrow}$ is just the inclusion relation $\subseteq$. In words, A is discernible from B in $\mathscr{P}_{op}(U)$ if and only if there exists $a \in A$ with $a \notin B$.

We now remark that, with that topology on $\mathscr{P}_{op}(U)$, the equation $U \cong \mathscr{P}_{op}(U)$ is *unsolvable* within $ALEX$. The syntactical first-order translation of this 'Cantor's theorem' for $ALEX$ is just the incompatibility of $(\mathscr{B})$ with $(\triangle)$ (cf. Theorem 4.5.4). Yet, the situation is by far less disastrous in $ALEX$ than in SET, for there are some (infinite) $\mathcal{U}$ here such that $|\mathscr{P}_{op}(U) \setminus \mathrm{rng}[\cdot]_{\mathcal{U}}| = 1$. This will be illustrated in Chapter 5 where we exhibit a solution to $U \cong \mathscr{P}_{op}(U)$ within $SCOTT$, the category of *Scott spaces*, which are just the dcpo's endowed with the Scott-topology. Notice that the specialization preorder of a Scott-space coincides with the ordering of the corresponding dcpo.

Remark 4.7.1. Another possible way to make $\mathscr{P}(U)$ into a Alexandroff space is by defining $A \lhd_{\mathscr{P}(U)} B$ if and only if $B \subseteq A^{\lhd_U}$, which may even seem more natural for then $U \longrightarrow \mathscr{P}(U) : u \longmapsto \{u\}$ is Alexandroff-continuous. The restriction of this to $\mathscr{P}_{op}(U)$ is now the reverse inclusion $\supseteq$, and that $U \cong \mathscr{P}_{op}(U)$ is still unsolvable within $ALEX$ follows from Proposition 4.5.5. Likewise, one can show that the equation $U \cong \mathscr{P}_{cl}(U)$, whether this latter is equipped with $\subseteq$ or $\supseteq$, is unsolvable within $ALEX$. Incidentally, a proof of this in the case where $\mathscr{P}_{cl}(U)$ is equipped with $\subseteq$ was originally given in [11], where it was shown that, given an *ordered* set U, there is no surjective monotone function $U \longrightarrow \mathscr{P}(U)^{\downarrow}$ (a second-order version of Lemma 4.5.3).

Quasi-discrete spaces

If the specialization preorder of an Alexandroff space is *symmetric*, that is to say, is an equivalence relation, we say that U is a *quasi-discrete space*. It is very easy to see that a topological space U is quasi-discrete if and only if $\mathscr{P}_{op}(U) = \mathscr{P}_{cl}(U)$. In words, the topology of a quasi-discrete space is generated by an equivalence relation: the open subsets, as well as the closed ones, are just unions of equivalence classes. We call the category of quasi-discrete spaces $QUASI$ (this is a sub-category of $ALEX$). According to Theorem 4.4.2, a model of $(\!(\triangle)\!) + (\mathscr{C})$ is just a solution $\mathcal{U}$ to $U \simeq \mathscr{P}_{op}(U)$ - equally $U \simeq \mathscr{P}_{cl}(U)$ - within $QUASI$; in such a $\mathcal{U} \doteq_{\mathcal{U}}$ coincides with $\lhd_U$.

Now, given a quasi-discrete space U, the appropriate way to enrol $\mathscr{P}(U)$ in $QUASI$ is by defining $A \lhd_{\mathscr{P}(U)} B$ if and only if $A \subseteq B^{\lhd_U}$ *and* $B \subseteq A^{\lhd_U}$, i.e., if and only if $A^{\lhd_U} = B^{\lhd_U}$, which is thus an equivalence relation on $\mathscr{P}(U)$. But the restriction of this to $\mathscr{P}_{op}(U)/\mathscr{P}_{cl}(U)$ is the identity, so that, for more obvious reasons here, there is no solution $\mathcal{U}$ to $U \cong \mathscr{P}_{op}(U)/U \cong \mathscr{P}_{cl}(U)$ within *QUASI*, for such a $\mathcal{U}$ would be a model of $(\triangle)+(\mathscr{C})+(x \doteq y \rightarrow x = y)$.

Remark 4.7.2. Referring back to Remark 2.3.1, it is easily seen that for a set-theoretic structure $\mathcal{U}$ in *QUASI*, demanding that $[\cdot]_{\mathcal{U}} : U \longrightarrow \mathscr{P}(U)$ be continuous actually amounts to demanding that $\lhd_U$ is a bisimulation.

Nevertheless, it is possible to define a *consistent* notion of coherence in models of $((\triangle)) + (\mathscr{C})$. We shall say that a solution $\mathcal{U}$ to $U \simeq \mathscr{P}_{op}(U)$ within *QUASI* is *acceptable* if for any indiscernible u, v in U, that is, with $u \lhd_U v$, either $[u]_{\mathcal{U}} = [v]_{\mathcal{U}} = \emptyset$ or $[u]_{\mathcal{U}} \cap [v]_{\mathcal{U}} \neq \emptyset$, which clearly vouches for a certain coherence in the process of assigning extensions to sets. The first-order translation of this condition on the axiomatic side is expressed by $(x \doteq y \rightarrow x \between y)$, where $x \between y :\equiv ((x = \Lambda \wedge y = \Lambda) \vee \exists z(z \in x \wedge z \in y))$.

At least, any acceptable solution is infinite, as the next observation shows:

Fact 4.7.1. Any model $\mathcal{U}$ of $(\Lambda) + (\mathscr{M}) + (\mathscr{C}) + (x \doteq y \rightarrow x \between y)$ is infinite.

Proof. First notice that $\Lambda \between \mathscr{M}(x)$ for no x, and that, as $\dot{\leqslant}$ coincides with $\doteq$ under $(\mathscr{C})$, we have $\mathscr{M}(x) \between \mathscr{M}(y) \Rightarrow x \doteq y$. Then it follows from the assumptions that we can define a potentially infinite sequence of elements in U by iterating $\mathscr{M}(\cdot)$ from $\Lambda^{\mathcal{U}}$ (as in Fact 4.5.1). ⊣

Just to stress the combinatoric nature of seeking an acceptable solution, we now formulate the corresponding version of Theorem 4.4.2.

Fact 4.7.2. A set U is the universe of an acceptable solution if and only if there is an equivalence R on U together with a bijection $f : U \longrightarrow \mathscr{P}(U/_R)$ such that $u\, R\, v \Rightarrow f`u \cap f`v \neq \emptyset$ or $f`u = f`v = \emptyset$.

Interestingly, the existence of an acceptable solution was established in [5], without any reference to Skala's set theory. The authors used that structure to promote what is called '*rough set theory*' (see also [6]). As furtively mentioned in 3.12, that solution, whose cardinality is $2^{2^{\aleph_0}}$, arises from an inverse limit construction in *DCPO*. Although we admit to not having made any serious attempt, it might be interesting to try to characterize those μ such that 2^μ is the cardinal of the universe of an acceptable solution. Call such a cardinal *acceptable*. If the construction given in [5] generalizes to κ-dcpo's for any regular cardinal κ - which seems to be the case - then 2^κ is acceptable. But perhaps there is a more direct way to generate acceptable solutions; and then, can one prove an acceptable version of Theorem 4.6.5?

Chapter 5

SKOLEM'S SPINE MODEL(S)

The twin models we present here first appeared in [37], and then in [38] where Skolem finally proved the consistency of $Comp[\mathscr{L}_*^+]$. We show in this chapter that these are in fact natural *topological* models of $Abst_{\circlearrowleft}[\mathscr{L}_{\tau*}^+]$, (re)establishing by the way the consistency of $Abst[\mathscr{L}_{\tau*}^+] + Ext$ (see [26]). We then start the comparison between $(\Diamond) + Abst[\mathscr{L}_{\tau*}^+]$ and $(\Diamond) + Comp[\mathscr{L}^{[+]}]$.

5.1 The $\mathscr{B}$-sequence

Any model of $Comp[\mathscr{L}_*^+]$ is a model of $(\Lambda) + (\mathrm{V}) + (\mathrm{W}) + (\mathscr{B})$, so we start by looking - as Skolem did - at sets which arise from Λ & V under $(\mathscr{B})$.

Assume $(\mathscr{B})$ and define inductively $\mathscr{B}^n(v)$, for $n \in \mathbb{N}$, as follows:

$$\begin{cases} \mathscr{B}^0(v) := v \\ \mathscr{B}^{n+1}(v) := \mathscr{B}(\mathscr{B}^n(v)). \end{cases}$$

It is then easily checked that, for each $n \in \mathbb{N}$,

$$\forall x \forall y (\mathscr{B}^n(x) \in \mathscr{B}^n(y) \leftrightarrow x \in y) \qquad (\dagger).$$

Now, let $\Lambda_n := \mathscr{B}^n(\Lambda)$ and $\mathrm{V}_n := \mathscr{B}^n(\mathrm{V})$, for any $n \in \mathbb{N}$.

Proposition 5.1.1. *Assuming* $(\Lambda) + (\mathrm{V}) + (\mathrm{W}) + (\mathscr{B})$, *we have*

(1) for any $m, n \in \mathbb{N}$, $\Lambda_m \notin \Lambda_n$ *and* $\mathrm{V}_m \in \mathrm{V}_n$ *;*

(2) for any $m, n \in \mathbb{N}$, $m < n \Leftrightarrow \Lambda_m \notin \mathrm{V}_n \Leftrightarrow \mathrm{V}_m \in \Lambda_n$ *;*

(3) for any $n \in \mathbb{N}$, $\Lambda_n \notin \mathrm{W}$ *and* $\mathrm{V}_n \in \mathrm{W}$ *;*

(4) for any $n \in \mathbb{N}$, $\mathrm{W} \notin \Lambda_n$ *and* $\mathrm{W} \in \mathrm{V}_n$ *.*

Proof. If $m \geqslant n$, it follows from (†) that $\mathscr{B}^m(x) \in \mathscr{B}^n(y) \leftrightarrow \mathscr{B}^{m-n}(x) \in y$, and if $m < n$, that $\mathscr{B}^m(x) \in \mathscr{B}^n(y) \leftrightarrow x \in \mathscr{B}^{n-m}(y) \leftrightarrow \mathscr{B}^{n-m-1}(y) \in x$. Now, to obtain (1), take respectively $x = y := \Lambda$ and $x = y := \mathrm{V}$; and to get (2), take $x := \Lambda$ and $y := \mathrm{V}$, or $x := \mathrm{V}$ and $y := \Lambda$. Notice that (3) follows directly from (1) & (2), and (4) follows from (3). ⊣

Corollary 5.1.2. *For any $n, m \in \mathbb{N}$, $\mathrm{V}_n \neq \Lambda_m$; and $\Lambda_n \neq \Lambda_m$, $\mathrm{V}_n \neq \mathrm{V}_m$ unless $n = m$. Consequently, any model of $(\mathscr{B})$ is infinite.*

Only the truth value of $\mathrm{W} \in \mathrm{W}$ has not been settled by Proposition 5.1.1, and this is really a matter of choice, as the next section shows.

5.2 The model(s)

Let U stand for $\{a_0, a_1, \ldots, a_n, \ldots, c, \ldots, b_n, \ldots, b_1, b_0\}$, where a_n is meant to be a name for Λ_n, b_n for V_n, and c for W. Then we define the membership relation on U in accordance with Proposition 5.1.1, i.e. $[a_n]_{\mathcal{U}} := \{b_m \mid m < n\}$ and $[b_n]_{\mathcal{U}} := U \setminus \{a_m \mid m < n\}$, for any $n \in \mathbb{N}$, and either $[c]_{\mathcal{U}} := \{b_n \mid n \in \mathbb{N}\}$ or $[c]_{\mathcal{U}} := \{b_n \mid n \in \mathbb{N}\} \cup \{c\}$, depending on whether we want $c \notin_{\mathcal{U}} c$ or $c \in_{\mathcal{U}} c$. Explicitly, here is the truth table of the membership relation(s) thus defined:

$\in_{\mathcal{U}}$	a_0	a_1	a_2	$\cdots$	c	$\cdots$	b_2	b_1	b_0
a_0	0	0	0	$\cdots$	0	$\cdots$	0	0	1
a_1	0	0	0	$\cdots$	0	$\cdots$	0	1	1
a_2	0	0	0	$\cdots$	0	$\cdots$	1	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$⋰$	$\vdots$	$\vdots$	$\vdots$
c	0	0	0	$\cdots$	0/1	$\cdots$	1	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$⋰$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
b_2	0	0	0	$\cdots$	1	$\cdots$	1	1	1
b_1	0	0	1	$\cdots$	1	$\cdots$	1	1	1
b_0	0	1	1	$\cdots$	1	$\cdots$	1	1	1

It is easily seen from this table that, for any $u, v \in U$, $u \leqslant_{\mathcal{U}} v \Leftrightarrow u \dot{\leqslant}_{\mathcal{U}} v \Leftrightarrow u \leqslant v$, where this latter is just the ordering of the elements as they appear:

$$a_0 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant c \leqslant \cdots \leqslant b_2 \leqslant b_1 \leqslant b_0 \,.$$

Equipped with that ordering, U is a *complete* chain and we observe that the collectable subsets in $\mathcal{U}$ are all the upper sets except $\{b_n \mid n \in \mathbb{N}\} \cup \{c\}$ when $c \notin_{\mathcal{U}} c$, or except $\{b_n \mid n \in \mathbb{N}\}$ when $c \in_{\mathcal{U}} c$, so that, in both cases, $\mathcal{U}$ just miss being a model of ((△)) (cf. Theorem 4.3.4).

As U is a complete lattice, it is in particular a *dcpo*, and so is its order dual U^*. With this in mind, we can see that the collectable subsets are exactly the *Scott-open* subsets of U when $c \notin_{\mathcal{U}} c$; whereas these are just the *Scott-closed* ones of U^* when $c \in_{\mathcal{U}} c$ (seeing that among all the upper sets in U, only $\{b_n \mid n \in \mathbb{N}\} \cup \{c\}$ is not Scott-open, and that among all the lower sets in U^*, only $\{b_n \mid n \in \mathbb{N}\}$ is not Scott-closed). Therefore, as the collectable subsets in $\mathcal{U}$ are the open/closed subsets of U with respect to some suitable topology on U, it will result that $\mathcal{U} \models (\!(\Box)\!)$ but $\not\models (\Diamond)$ when $c \notin_{\mathcal{U}} c$, whereas $\mathcal{U} \models (\!(\Diamond)\!)$ but $\not\models (\Box)$ when $c \in_{\mathcal{U}} c$ (see Corollary 5.3.3 below).

But the key observation here is that $\mathscr{B}(c) = c$ in $\mathcal{U}$, from which it follows that $\mathcal{U} \models (\Lambda) + (\mathrm{V}) + (\mathrm{W}) + (\mathscr{B})$, as expected. In fact, it was shown in [37, 38] that $\mathcal{U}$ is a model of $Comp[\mathscr{L}_*^+]$ - and so the simplest conceivable. In the next section we improve Skolem's result by showing that, even more surprisingly, $\mathcal{U} \models Abst_{\circlearrowleft}[\mathscr{L}_{\tau*}^+]$.

It should have been noticed before that by interchanging 0 with 1 everywhere in the truth-table of $\in$, it is manifest that $\mathcal{U}$ with $c \in_{\mathcal{U}} c$ is just isomorphic to the dual $\mathcal{U}^{\mathrm{c}}$ of $\mathcal{U}$ with $c \notin_{\mathcal{U}} c$, so that we may actually get rid of the schizophrenic nature of $\mathcal{U}$. In the next section we shall only consider the case $\epsilon_{\mathcal{U}}`(c, c) = 0$ and formally adopt the valued setting. It is worth recalling at this point the few things on Scott-continuity we mentioned in 3.11.

5.3 The proof

A simple look at the table shows that $\epsilon_{\mathcal{U}}$ is monotone. We have much more:

Proposition 5.3.1. *$\epsilon_{\mathcal{U}} : U \times U \longrightarrow 2$ is Scott-continuous.*

Proof. It suffices to show that $\epsilon_{\mathcal{U}}$ is continuous in each variable separately. Let $v \in U$ and $\emptyset \neq D \subseteq U$. Suppose first that $\bigvee D \in D$. Then we have $\epsilon_{\mathcal{U}}`(\bigvee D, v) \leqslant \bigvee_{d \in D} \epsilon_{\mathcal{U}}`(d, v) \leqslant \epsilon_{\mathcal{U}}`(\bigvee D, v)$, and likewise we show that $\epsilon_{\mathcal{U}}`(v, \bigvee D) = \bigvee_{d \in D} \epsilon_{\mathcal{U}}`(v, d)$. Now suppose $\bigvee D \notin D$. So $\bigvee D = c$ and $D \subseteq \{a_n \mid n \in \mathbb{N}\}$. If $v = a_m$ for some $m$, we have $\epsilon_{\mathcal{U}}`(d, v) = \epsilon_{\mathcal{U}}`(v, d) = 0$, for all $d \in D$, so that $\bigvee_{d \in D} \epsilon_{\mathcal{U}}`(d, v) = \bigvee_{d \in D} \epsilon_{\mathcal{U}}`(v, d) = 0 = \epsilon_{\mathcal{U}}`(\bigvee D, v) = \epsilon_{\mathcal{U}}`(v, \bigvee D)$. If $v = b_m$ for some $m$, as there always exists $a_n \in D$ with $n > m$, we have $\epsilon_{\mathcal{U}}`(d, v) = \epsilon_{\mathcal{U}}`(v, d) = 1$, for some $d \in D$, so that $\bigvee_{d \in D} \epsilon_{\mathcal{U}}`(d, v) = \bigvee_{d \in D} \epsilon_{\mathcal{U}}`(v, d) = 1 = \epsilon_{\mathcal{U}}`(\bigvee D, v) = \epsilon_{\mathcal{U}}`(v, \bigvee D)$. If $v = c$, we have $\epsilon_{\mathcal{U}}`(d, v) = \epsilon_{\mathcal{U}}`(v, d) = 0$, for all $d \in D$, so that $\bigvee_{d \in D} \epsilon_{\mathcal{U}}`(d, v) = \bigvee_{d \in D} \epsilon_{\mathcal{U}}`(v, d) = 0 = \epsilon_{\mathcal{U}}`(\bigvee D, v) = \epsilon_{\mathcal{U}}`(v, \bigvee D)$ because $\epsilon_{\mathcal{U}}`(c, c) = 0$ precisely. ⊣

Corollary 5.3.2. *$[\![\cdot]\!]_{\mathcal{U}} : U \longrightarrow \langle U \to 2 \rangle$ is a Scott-homeomorphism.*

Proof. That $[\![\cdot]\!]_{\mathcal{U}} : U \longrightarrow \langle U \to 2\rangle$ and is Scott-continuous directly follow from Proposition 5.3.1. As was initially observed, the collectable subsets in $\mathcal{U}$ are exactly the Scott-open subsets of U, which shows that $[\![\cdot]\!]_{\mathcal{U}}$ is onto. Now, that its inverse $[\![\cdot]\!]_{\mathcal{U}}^{-1}$ is also Scott-continuous is easily seen. ⊣

Corollary 5.3.3. *$\mathcal{U} \models ((\Box))$ but $\not\models (\Diamond)$.*

Proof. We already know that $\mathcal{U} \models ((\Box))$. To show that $\mathcal{U} \not\models (\Diamond)$, it suffices to notice that $\bigcap`\{z \mid z \in z\}$' does not exist in $\mathcal{U}$. Indeed, if this latter was to exist, it should be c. But as $\mathcal{U} \models \forall z(z \in z \to c \in z)$, we should then have $c \in_{\mathcal{U}} c$, which is precisely false here. ⊣

Remark 5.3.1. By the way, Corollary 5.3.3 provides a counter-example to the converse of Fact 4.1.1, since we noticed that $\langle U; \dot{\leqslant}_{\mathcal{U}}\rangle$ is a complete lattice.

We are now ready to proof the main result:

Theorem 5.3.4. *$\mathcal{U} \models Abst_{\circlearrowleft}[\mathscr{L}^{+}_{\tau*}]$.*

Proof. We show by induction on the complexity that each term $\tau(\bar{p})$ of $\mathscr{L}_\tau(U)$ has a 'suitable' Scott-continuous interpretation $\tau^{\mathcal{U}} : (\bar{u}) \longmapsto \tau^{\mathcal{U}}(\bar{u})$.

First, if τ is just a variable, say p_k in $\bar{p}$, then we take $\tau^{\mathcal{U}} : (\bar{u}) \longmapsto u_k$, which is clearly Scott-continuous; and if τ is any fixed $v \in U$, then we take $\tau^{\mathcal{U}} : (\bar{u}) \longmapsto v$, which is also obviously Scott-continuous.

We now turn to the case where $\tau(\bar{p})$ is a set abstract $\lambda x\, \varphi$ for a $\mathscr{L}_\tau(U)$-formula $\varphi(x, \bar{p})$. Here, that the interpretation is 'suitable' means, of course, that $|\tau^{\mathcal{U}}(\bar{u}) \cdot v|_{\mathcal{U}} = |\varphi(v, \bar{u})|_{\mathcal{U}}$ for any $\bar{u}, v$ in U, from which incidentally results the uniqueness of such a suitable interpretation.

The proof goes by induction on the complexity of φ:

1) φ is atomic, say φ is $\rho \cdot \sigma$ in which $\rho(x, \bar{p}), \sigma(x, \bar{p})$ are $\mathscr{L}_\tau(U)$-terms. Then $\tau^{\mathcal{U}} : (\bar{u}) \longmapsto [\![v \mapsto \epsilon_{\mathcal{U}}`(\sigma^{\mathcal{U}}(v, \bar{u}), \rho^{\mathcal{U}}(v, \bar{u}))]\!]_{\mathcal{U}}^{-1}$ is clearly Scott-continuous, and this is the suitable interpretation of $\lambda x\, \varphi$.

2) $\varphi(x, \bar{p})$ is $\psi(x, \bar{p}) \vee \chi(x, \bar{p})$. Let $\sigma(\bar{p})$ stand for $\lambda x\, \psi$ and $\rho(\bar{p})$ for $\lambda x\, \chi$. Then $\tau^{\mathcal{U}} : (\bar{u}) \longmapsto [\![v \mapsto \vee`(\sigma^{\mathcal{U}}(\bar{u}) \cdot v\,,\, \rho^{\mathcal{U}}(\bar{u}) \cdot v)]\!]_{\mathcal{U}}^{-1}$ is the suitable interpretation of $\lambda x\, \varphi$; it is Scott-continuous for so is $\vee : 2 \times 2 \to 2 : (x, y) \mapsto \max\{x, y\}$.

3) $\varphi(x, \bar{p})$ is $\psi(x, \bar{p}) \wedge \chi(x, \bar{p})$. Likewise with $\wedge : (x, y) \mapsto \min\{x, y\}$.

4) $\varphi(x, \bar{p})$ is $\exists y \psi(x, y, \bar{p})$. Let $\sigma(y, \bar{p})$ stand for $\lambda x\, \psi$. Notice that for any $\bar{u}, v$ in U, $|\exists y \psi(v, y, \bar{u})|_{\mathcal{U}} = \max_{w \in U} |\psi(v, w, \bar{u})|_{\mathcal{U}} = \max_{w \in U} |\sigma^{\mathcal{U}}(w, \bar{u}) \cdot v|_{\mathcal{U}} = |\sigma^{\mathcal{U}}(b_0, \bar{u}) \cdot v|_{\mathcal{U}}$. Thence $\tau^{\mathcal{U}} : (\bar{u}) \longmapsto [\![v \mapsto \sigma^{\mathcal{U}}(b_0, \bar{u}) \cdot v]\!]_{\mathcal{U}}^{-1}$ is the suitable Scott-continuous interpretation of $\lambda x\, \varphi$.

5) $\varphi(x, \bar{p})$ is $\forall y \psi(x, y, \bar{p})$. Likewise, for any $v, \bar{u}$, we have $|\forall y \psi(v, y, \bar{u})|_{\mathcal{U}}$ $= \min_{w \in U} |\psi(v, w, \bar{u})|_{\mathcal{U}} = \min_{w \in U} |\sigma^{\mathcal{U}}(w, \bar{u}) \cdot v|_{\mathcal{U}} = |\sigma^{\mathcal{U}}(a_0, \bar{u}) \cdot v|_{\mathcal{U}}$, so that $\tau^{\mathcal{U}} : (\bar{u}) \longmapsto \sigma^{\mathcal{U}}(a_0, \bar{u})$ is the suitable Scott-continuous interpretation of $\lambda x\, \varphi$.

Finally, we consider the case where τ is a reflexive set abstract $\lambda_y x\, \varphi$ for a $\mathscr{L}_\tau(U)$-formula $\varphi(x, y, \bar{p})$. Let $\sigma^U(y, \bar{p})$ be the suitable interpretation for $\lambda x\, \varphi$, and then, given $\bar{u}$ in U, let $f_{\bar{u}} : U \longrightarrow U : v \longmapsto \sigma^{\mathcal{U}}(v, \bar{u})$. Clearly $f_{\bar{u}}$ is Scott-continuous on U; then let $\mu(f_{\bar{u}})$ be its least fixpoint. As the application $(\bar{u}) \longmapsto f_{\bar{u}}$ is Scott-continuous, so is $(\bar{u}) \longmapsto \mu(f_{\bar{u}})$; and as we have $|\mu(f_{\bar{u}}) \cdot v|_{\mathcal{U}} = |\sigma^{\mathcal{U}}(\mu(f_{\bar{u}}), \bar{u}) \cdot v|_{\mathcal{U}} = |\varphi(v, \mu(f_{\bar{u}}), \bar{u})|_{\mathcal{U}}$, it follows that $\tau^{\mathcal{U}} : (\bar{u}) \longmapsto \mu(f_{\bar{u}})$ is a suitable Scott-continuous interpretation of $\lambda_y x\, \varphi$ (note that such a suitable interpretation is not necessarily unique here).

To have a clear conscience, it would remain to convince the reader that the interpretation of the abstractor we have given fulfils the substitutivity clause stated in Section 2.3, which is particularly awkward to check in details. $\dashv$

Remark 5.3.2. According to Corollary 5.3.2, the spine model $\mathcal{U}$ with $c \notin_{\mathcal{U}} c$ is a solution to $U \cong \langle U \to 2 \rangle$ in *DCPO*. But using the machinery of dcpo's, we know that the minimal solution to this reflexive equation can be obtained by iterating the functor $\langle \cdot \to 2 \rangle$ and then taking the inverse limit of the sequence so generated. Evidently this minimal solution should be $\mathcal{U}$. Likewise, the spine model $\mathcal{U}$ with $c \in_{\mathcal{U}} c$ is easily seen to be the minimal solution to $U \cong \langle U \to 2^* \rangle$. Of course, as ordered sets, these are isomorphic, but the homeomorphisms - and so the set-theoretic structures - slightly differ. It is remarkable that these canonical solutions arise from pure and abstract set-theoretic considerations.

5.4 Abstraction versus comprehension

First of all, since $\mathcal{U}$ is obviously extensional, it should be stressed that we thus have also established the consistency of $Abst[\mathscr{L}_{\tau*}^+] + Ext$. The original proof of this, using a term model construction, was by far more complicated. It is described in the next chapter where the theory $Abst[\mathscr{L}_{\tau*}^+] + Ext$ is still further broken down. In what follows we try to compare $(\diamond) + Abst[\mathscr{L}_{\tau*}^+]$, the consistency of which has just been proved, with $GPK^+ \equiv (\diamond) + Comp[\mathscr{L}^{[+]}]$.

We start the comparison on the semantic side. Let S_ω stand in what follows for the spine model with $c \in_{\mathcal{U}} c$.

We have shown that $S_\omega \models (\diamond) + Abst[\mathscr{L}_{\tau*}^+]$, and this is clearly the simplest model. As a topological space, S_ω appears as the canonical solution to $U \cong \mathscr{P}_{cl}(U)$ within the category of Scott spaces. On the other hand, the hyperuniverse N_ω, which is known to be a model of $(\Diamond)+Comp[\mathscr{L}^{[+]}]$, is also the canonical solution to $U \cong \mathscr{P}_{cl}(U)$ but within the category of complete metric spaces; these are T_2-spaces, whereas Scott spaces are only T_0-spaces. It is also worth noting that $|N_\omega| = 2^{\aleph_0}$ whereas $|S_\omega| = \aleph_0$.

For $\exists$ to be continuous (in the sense of Chapter 4), the compactness of N_ω is the key property - see [17]. In S_ω, it is monotonicity - cf. 4) in the proof of Theorem 5.3.4. By the way, in that proof, to show that reflexive set abstracts are naturally interpretable in S_ω, we appealed to the fixpoint property for Scott-continuous maps $S_\omega \longrightarrow S_\omega$. We do not know whether it would be possible by a similar argument to prove that $N_\omega \models Comp_\circlearrowleft[\mathscr{L}^+]$. Note that it has been shown that $N_\omega \models Comp_\circlearrowleft[x = y]$, or even that $N_\omega \models Comp_\circlearrowleft[x = y \vee x = p]$, etc. But, as far as we know, the consistency of $Comp_\circlearrowleft[\mathscr{L}^+]$ has not been established.

On the axiomatic side, the first thing to say is that $(\Diamond) + Abst[\mathscr{L}^+_{\tau*}]$ and $(\Diamond)+Comp[\mathscr{L}^{[+]}]$ are strongly incompatible (assuming Ext as always). This is, of course, a straightforward consequence of the following observation.

Fact 5.4.1. $(\mathscr{A})$ is incompatible with $Abst[\mathscr{L}^+_{\tau*}] + Ext$.

Proof. This follows from Proposition 4.5.6, but we give here a more explicit proof. Let $\tau(p)$ be $\{x \mid \{z \mid z \in x\} \in p\}$ and then let r stand for $\tau(\mathscr{A}(\Lambda))$. Assuming $(\mathscr{A})$ and $Abst[\mathscr{L}^+_{\tau*}]$, r is a set, and now, using Ext as in the proof of Fact 3.8.1, it is easily seen that $r \in r \leftrightarrow r \notin r$. ⊣

It is not known whether the use of parameters in set abstracts could be avoided in the proof. Thus far, the best we are able to show is the following.

Proposition 5.4.1. *$Abst[\mathscr{L}^+_{\tau*}]'$ and $(\Diamond)' + Comp[\mathscr{L}^{[+]}]$ are incompatible, where $Abst[\mathscr{L}^+_{\tau*}]'$ and $(\Diamond)'$ respectively gather those parameter-free instances of $Abst[\mathscr{L}^+_{\tau*}]$ and $(\Diamond)$.*

Although not very difficult, the proof of Proposition 5.4.1 requires some basic features of the theory $(\Diamond)' + Comp[\mathscr{L}^{[+]}]$, which was called GPK for historical reasons and was deeply investigated in [12]. Referring to this latter, we shall content ourselves hereafter with pointing out what we need.

In GPK, any class $X :=$ '$\{x \mid \varphi(x)\}$' definable without parameters has a 'closure', namely $\{x \mid_\diamond \varphi\}$, which we denote here by $\overline{X}$. Thereupon it was shown that many pseudo-topological considerations can be developed within GPK. Thus, it was proved that Von Neumann ordinals are definable in GPK and that these are isolated points, that is, '$\{\alpha\}$' is closed for each α in On, the class of ordinals, which in turn was proved to satisfy $\overline{On} = On \cup \{\overline{On}\}$ (in response to Burali-Forti's paradox). These facts will be enough to prove Proposition 5.4.1 by showing that the class $X :=$ '$\{x \mid \{z \mid x \in x\} \in x\}$', interpreted in the obvious way, cannot be closed in GPK, whereas this should be a set according to $Abst[\mathscr{L}_{\tau *}^{+}]'$.

Proof. We work in GPK and rewrite the class X as

$$`\{x \mid (x \in x \wedge \mathrm{V} \in x) \vee (x \notin x \wedge \Lambda \in x)\}' .$$

Notice that $On \setminus \{\Lambda\} \subseteq X$. We mentioned that $\overline{On}$ is an accumulation point of On, and that Λ is an isolated point. It is then easy to see that $\overline{On}$ is also an accumulation point of $On \setminus \{\Lambda\}$, and so of X. Whence $\overline{On} \in \overline{X}$. But $\overline{On}$ does not belong to X, for $\overline{On} \in \overline{On}$ and it is clear that $\mathrm{V} \notin \overline{On} (= On \cup \{\overline{On}\})$. Therefore X is not closed. ⊣

We have thus shown that GPK is allergic to such a typical set abstract as $\{x \mid \{z \mid x \in x\} \in x\}$. We do not know whether $Abst[\mathscr{L}_{\tau *}^{+}]' + Comp[\mathscr{L}^{[+]}]$ is already inconsistent (assuming Ext). Note that such results may serve to measure the amount of negations that appear in eliminating *positive* set abstracts. Proposition 5.4.1 is due to Hinnion and appeared in [24].

The existence of W is emblematic of all these positive set-theories. The schizophrenic nature of the spine model(s) revealed that $\mathrm{W} \in \mathrm{W}$ is undecidable from $Abst[\mathscr{L}_{\tau *}^{+}]$ - and so from $Comp[\mathscr{L}_{\tau *}^{+}]$. Although this is very unlikely to be true, we do not know whether $\mathrm{W} \in \mathrm{W}$ is decidable from $(\Diamond)+Abst[\mathscr{L}_{\tau *}^{+}]$ / $(\Box)+Abst[\mathscr{L}_{\tau *}^{+}]$; as well as it is not known whether $\mathrm{W} \in \mathrm{W}$ is decidable from GPK^{+}, i.e., $(\Diamond)+Comp[\mathscr{L}^{[+]}]$. It seems nobody has looked into the status of $\mathrm{W} \in \mathrm{W}$ in hyperuniverses.

Chapter 6

TERM MODELS

This last chapter discusses the term model construction that was used in [24] to prove the consistency of $Abst[\mathscr{L}_{\tau*}^{+}] + Ext$. In initiating the comparison between the term models so constructed and the spine models of Chapter 5, we show that $Abst[\mathscr{L}_{\tau*}^{+}] + Ext$ has quantifier elimination.

6.1 Exploring a syntactical universe

By a *term model* we mean any set-theoretic structure $\mathcal{U}$ whose universe U is exclusively made of *terms* - constants, variables, set abstracts - conceived as *syntactical expressions* (e.g. sequences of symbols in the meta-theory), and such that for every $\{x \mid \varphi\} \in U$, we have $\mathcal{U} \models \forall x(x \in \{x \mid \varphi\} \leftrightarrow \varphi)$.

In this chapter U is taken to be the set of all *closed* (i.e. parameter-free) set abstracts $\{x \mid \varphi\}$, where $\varphi(x)$ is first in $\mathscr{L}_{\tau}{}^{+}$, and then we will restrict ourselves to $\mathscr{L}_{\tau*}^{+}$-formulas. The letters $a, b, c, \ldots$ will stand for elements in U.

We notice that there is a canonical interpretation of the abstractor on U. Namely, given any $\mathscr{L}_{\tau}{}^{+}$-formula $\varphi(x, \bar{p})$ and any $\bar{a}$ in U of the same length as $\bar{p}$, we just take $\{x \mid \varphi\}^{\mathcal{U}}(\bar{a})$ to be $\{x \mid \psi\}$ where ψ is $\varphi(x, \bar{a})$. It is clear that this interpretation fulfils the substitutivity property stated in 2.3, in which $=$ is taken to be the identity on U. Unless otherwise mentioned, this latter is assumed to be the interpretation of the equality relation on U - though we will see below (Fact 6.1.2) that it is not suitable at all. Thus, in order to turn U into an $\mathscr{L}_{\tau}$-structure, we only have to specify the interpretation of $\in$.

For convenience, we may identify in what follows any structure $\mathcal{U}$ on U with a subset A of $U \times U$, namely $\in_{\mathcal{U}}$, and then write $A \models \varphi$ for $\mathcal{U} \models \varphi$, $|\varphi|_A$ for $|\varphi|_{\mathcal{U}}$, and so on. We now show how to select those A satisfying $Abst[\mathscr{L}_{\tau}{}^{+}]$.

To proceed, for any $A \subseteq U \times U$, we define A^{+}, the 'upgraded version' of A, as follows:

$$\text{for any } a, b \in U \text{ with } b = \{x \mid \varphi\}, \quad |a \in b|_{A^{+}} := |\varphi(a)|_{A}$$

The relevance of considering the upgrading operator $(\cdot)^+$ on $\mathscr{P}(U \times U)$ lies in the following observation.

Proposition 6.1.1. *$A \models Abst[\mathscr{L}_\tau{}^+]$ if and only if $A \in Fix((\cdot)^+)$.*

Proof.
Necessity: Let a, b in U with $b = \{x \mid \varphi\}$. We have $|a \in b|_{A^+} = |\varphi(a)|_A = |a \in b|_A$ (because $A \models Abst[\mathscr{L}_\tau{}^+]$), which shows that $A^+ = A$.
Sufficiency: Let $\varphi(x, \bar{p})$ be any $\mathscr{L}_\tau{}^+$-formula. Then take any $\bar{b}$ in U of the same length as $\bar{p}$, and let $\psi(x)$ stand for the $\mathscr{L}_\tau{}^+$-formula $\varphi(x, \bar{b})$. Assume $A = A^+$. It follows that, for all $a \in U$, $|a \in \{x \mid \varphi\}^A(\bar{b})|_A = |a \in \{x \mid \psi\}\,|_A = |a \in \{x \mid \psi\}\,|_{A^+} = |\psi(a)|_A = |\varphi(a, \bar{b})|_A$, showing that $A \models Abst[\varphi(x)]$. ⊣

Now, as $\mathscr{P}(U \times U)$ is a complete lattice, we may invoke the Knaster-Tarski theorem to show that $Fix((\cdot)^+) \neq \emptyset$, because:

Fact 6.1.1. $(\cdot)^+ : \mathscr{P}(U \times U) \longrightarrow \mathscr{P}(U \times U)$ is monotone.

Proof. Suppose $A \subseteq B$. Then the identity function $x \mapsto x$ defines a surjective $\mathscr{L}_\tau$-homomorphism from $\langle U; A \rangle$ onto $\langle U; B \rangle$. Now, by virtue of the preservation property of $\mathscr{L}_\tau{}^+$-formulas (cf. 2.3), it follows that $|\varphi(a)|_A \leqslant |\varphi(a)|_B$, for every $\mathscr{L}_\tau{}^+$-formula $\varphi(x)$ and any $a \in U$, which shows that $A^+ \subseteq B^+$. ⊣

We thus have easily established the consistency of $Abst[\mathscr{L}_\tau{}^+]$. But according to Fact 3.8.1, it is hopeless to try to find any model satisfying *Ext*.

Nevertheless, it was proved in Chapter 5 that $Abst[\mathscr{L}_{\tau*}^+] + Ext$ is consistent. As said therein, the original proof of this appealed to a term model construction, on which we shall now elaborate.

We therefore have to drop equality in formulas defining sets, and in what remains of this chapter U will stand for the set of all closed $\mathscr{L}_{\tau*}^+$-set abstracts. Notice that Proposition 6.1.1 remains true, so we are looking for *extensional* fixpoints of $(\cdot)^+$. As the following observation shows, we also have to rethink the interpretation of equality in term models.

Fact 6.1.2. There is no *normal* term model of $Abst[\mathscr{L}_{\tau*}^+] + Ext$.

Proof. Clearly, $Abst[\mathscr{L}_{\tau*}^+] + Ext \vdash \{x \mid x \in x\} = \{x \mid x \in x \wedge x \in x\}$. But as these set abstracts are *syntactically* different, they will differ from each other in any *normal* term model (because of our definition of what a term model is). ⊣

Anyway, in any structure fulfilling *Ext* the interpretation of $=$ does coincide with the one of $\doteq$, and we are going to see that in some term models of $Abst[\mathscr{L}_{\tau*}^+]$ this latter may indeed be taken to be $=$. So we may here and now assert the main result of this chapter.

Theorem 6.1.2. *There exists $B \in Fix((\cdot)^+)$ such that $B \models {}^\flat Ext$.*

The proof is given in the next section (see Theorem 6.2.4). But first, to comment on this, it is worth looking further into the structure of $Fix((\cdot)^+)$.

For any $A \subseteq U \times U$, we define $[A[:= \{B \subseteq U \times U \mid A \subseteq B\}$. Obviously, this is a complete sub-lattice of $\mathscr{P}(U \times U)$. Notice that $A^+ \in [A[$ if and only if for all $B \in [A[\,, B^+ \in [A[$. Then, for any $A \subseteq U \times U$ with $A^+ \in [A[$, let $(\cdot)^+_A : [A[\longrightarrow [A[$ denote the restriction of $(\cdot)^+$ on $[A[$. By the Knaster-Tarski theorem, we know that $Fix((\cdot)^+_A)$ is a complete lattice. Let $A_\star$ and $A^\star$ respectively stand for its least element and its greatest one. Obviously, $A^\star = \emptyset^\star$ for all $A \subseteq U$; and it is important to remember that $A_\star$ can be obtained *inductively* by iterating $(\cdot)^+$ from A.

We thus have shown that any $A \subseteq U \times U$ with $A \subseteq A^+$ can iteratively be extended to a model A_* of $Abst[\mathscr{L}^+_{\tau *}]$. Notice that $\emptyset^\star$ can also be obtained inductively by iterating $(\cdot)^+$ from $U \times U$; this is because $(U \times U)^+ \subseteq U \times U$ - or simply by *duality*: $A \longmapsto U \times U \setminus A$.

We are going to prove in the next section that $\emptyset_\star \models {}^\flat Ext$ and $\emptyset^\star \models {}^\flat Ext$, and in fact these are the only *extensional* term models of $Abst[\mathscr{L}^+_{\tau *}]$ we know. On the other hand, it is fairly easy to concoct *non-extensional* term models.

Theorem 6.1.3. *There exists $B \in Fix((\cdot)^+)$ such that $B \not\models {}^\flat Ext$.*

Proof. Recall that $\mathrm{W} = \{x \mid x \in x\}$ and let W' stand for $\{x \mid x \in x \wedge x \in x\}$. It is easy to see that $|\mathrm{W} \in \mathrm{W}|_{\emptyset_\star} = |\mathrm{W}' \in \mathrm{W}'|_{\emptyset_\star} = 0$ (use the fact that $\emptyset_\star$ is obtained inductively by iterating $(\cdot)^+$ from $\emptyset$). We now define a new structure A as follows:

$$\begin{cases} |a \in b|_A := |a \in b|_{\emptyset_\star} & \text{for all } a, b \in U \text{ such that } a \neq \mathrm{W}' \text{ or } b \neq \mathrm{W}' \\ |\mathrm{W}' \in \mathrm{W}'|_A := 1 \end{cases}$$

Clearly, $\emptyset_\star \subseteq A$, from which we show that $A \subseteq A^+$. For if $a \neq \mathrm{W}'$ or $b = \{x \mid \varphi\} \neq \mathrm{W}'$, we have $|a \in b|_A = |a \in b|_{\emptyset_\star} = |\varphi(a)|_{\emptyset_\star} \leqslant |\varphi(a)|_A = |a \in b|_{A^+}$; and $|\mathrm{W}' \in \mathrm{W}'|_{A^+} = |\mathrm{W}' \in \mathrm{W}' \wedge \mathrm{W}' \in \mathrm{W}'|_A = 1$. Now, we may take $B := A_\star$. Indeed, $|a \in \mathrm{W}|_{A_\star} = |a \in a|_{A_\star} = |a \in a \wedge a \in a|_{A_\star} = |a \in \mathrm{W}'|_{A_\star}$, for any $a \in U$, so that $A_\star \models \mathrm{W} \doteq \mathrm{W}'$; but $|\mathrm{W}' \in \mathrm{W}|_{A_\star} = |\mathrm{W}' \in \mathrm{W}'|_{A_\star} = 1$, whereas $|\mathrm{W} \in \mathrm{W}|_{A_\star} = 0$ (again use the fact that $A_\star$ is obtained inductively by iterating $(\cdot)^+$ from A, and $|\mathrm{W} \in \mathrm{W}|_A = 0$), showing that $A_\star \models \mathrm{W} \neq \mathrm{W}'$. $\dashv$

6.2 The proof

This section is devoted to proving Theorem 6.1.2. As announced, we show that $\emptyset_\star \models {}^\flat Ext$ and $\emptyset^\star \models {}^\flat Ext$. The proof is based on the fact that these fixpoints can be obtained inductively by iterating the upgrading operator. Clearly, by duality, we may concentrate on establishing the result for $\emptyset_\star$ only. To achieve this, we need some preliminary definitions and a few lemmas.

We first make explicit the transfinite sequence of iterates that leads to $\emptyset_\star$:

$$\begin{cases} \in_0 := \emptyset \\ \in_{\alpha+1} := (\in_\alpha)^+ \\ \in_\lambda := \bigcup_{\alpha<\lambda} \in_\alpha \qquad (\lambda \text{ limit}). \end{cases}$$

Thus we have $\emptyset_* = \in_\delta$, where δ is the least ordinal α such that $\in_{\alpha+1} = \in_\alpha$.

Given a $\mathscr{L}_\tau$-formula φ, we define a *primitive of* φ to be any 'maximal' atomic sub-formula of φ ('maximal' with respect to the relation 'is a sub-formula of'), and we denote the set of its primitives by $\mathcal{P}(\varphi)$. So φ can be built up from the formulas in $\mathcal{P}(\varphi)$ without using the abstractor $\{\cdot \mid -\}$, only by means of logical connectives and quantifiers.

In what remains of this section, all the formulas we shall be considering are assumed to be closed $\mathscr{L}^+_{\tau*}$-formulas φ with $\in_\delta \models \varphi$, unless otherwise explicitly stated. For such a formula φ, we let α_φ stand for the least ordinal α such that $\in_\alpha \models \varphi$. Notice that $\in_\alpha \models \varphi$ for all $\alpha \geqslant \alpha_\varphi$. We now define

$$\mathcal{D}(\varphi) := \{\psi(\bar{a}) \mid \psi(\bar{p}) \in \mathcal{P}(\varphi),\ \bar{a} \text{ in } U, \text{ and } \in_{\alpha_\varphi} \models \psi(\bar{a})\}$$

Given such a formula φ and $a \in U$, we are going to be interested in some specified occurrences of a (as sub-term) in φ. For that purpose, we conveniently use the notation $\varphi\langle a\rangle$ for the formula φ *together with* a given *coloring* of specified occurrences of a in it (possibly none). Then, given $b \in U$, $\varphi\langle b/a\rangle$ will stand for the formula obtained from φ by substituting b for each *colored* occurrence of a in $\varphi\langle a\rangle$.

We now move on to the key definition of *determining set.*

Definition. A *determining set* $\mathcal{D}$ *of degree* $\leq \alpha$ *for* $\varphi\langle a\rangle$ is a non-empty set of *atomic* formulas $\psi\langle a\rangle$, such that :

$$\begin{cases} \text{(i)} \ \in_\alpha \models \psi\langle a\rangle \ ; \\ \text{(ii) for each } b \in U, \\ \qquad \text{if } \in_\delta \models \psi\langle b/a\rangle \text{ for all } \psi\langle a\rangle \in \mathcal{D}, \text{ then } \in_\delta \models \varphi\langle b/a\rangle\,. \end{cases}$$

To begin with, we prove that any formula $\varphi\langle a\rangle$ has a determining set. Indeed, let $\mathcal{D}(\varphi\langle a\rangle) := \{\psi\langle a\rangle \mid \psi \in \mathcal{D}(\varphi)\}$ where, for each $\psi \in \mathcal{D}(\varphi)$, the colored occurrences of 'a' in $\psi\langle a\rangle$ are those occurrences of 'a' that are in the primitive of φ from which ψ is obtained, and that are colored in $\varphi\langle a\rangle$ (so it is understood that the hypothetical occurrences of 'a' substituted in that primitive are not colored).

Lemma 6.2.1. *$\mathcal{D}(\varphi\langle a\rangle)$ is a determining set of degree $\leq \alpha_\varphi$ for $\varphi\langle a\rangle$.*

Proof. The proof goes by induction on the complexity of φ.

- If φ is any atomic $\mathscr{L}^+_{\tau*}$-formula, then $\mathcal{D}(\varphi\langle a\rangle) = \{\varphi\langle a\rangle\}$, and that is obvious.

- If φ is $\varphi_1 \wedge \varphi_2$, we have $\alpha_\varphi = \max\{\alpha_{\varphi_1}, \alpha_{\varphi_2}\}$ and then $\mathcal{D}(\varphi_1\langle a\rangle) \cup \mathcal{D}(\varphi_2\langle a\rangle) \subseteq \mathcal{D}(\varphi\langle a\rangle)$ (where $\varphi_1\langle a\rangle$ and $\varphi_2\langle a\rangle$ are denoting respectively φ_1 and φ_2 with the coloring induced by the one of $\varphi\langle a\rangle$). Therefore, assuming $\in_\delta \models \psi\langle b/a\rangle$ for all $\psi\langle a\rangle \in \mathcal{D}(\varphi\langle a\rangle)$, we get $\in_\delta \models \varphi_1\langle b/a\rangle$ and $\in_\delta \models \varphi_2\langle b/a\rangle$ by the induction hypothesis, and thus $\in_\delta \models \varphi\langle b/a\rangle$.

- If φ is $\varphi_1 \vee \varphi_2$, then $\alpha_\varphi := \min\{\alpha_{\varphi_1}, \alpha_{\varphi_2}\}$. Assuming this latter to be α_{φ_1}, we have $\mathcal{D}(\varphi_1\langle a\rangle) \subseteq \mathcal{D}(\varphi\langle a\rangle)$. We then proceed as above.

- If φ is $\forall x\varphi'(x)$, then, for any $c \in U$, we have $\alpha_{\varphi'(c)} \leqslant \alpha_\varphi$ and so $\mathcal{D}(\varphi'(c)\langle a\rangle) \subseteq \mathcal{D}(\varphi\langle a\rangle)$ (where $\varphi'(c)\langle a\rangle$ is denoting $\varphi'(c)$ with the coloring induced on $\varphi'(x)$ by the one of $\varphi\langle a\rangle$; particularly when $c = a$, that occurrence of 'a' is not colored). Hence, assuming $\in_\delta \models \psi\langle b/a\rangle$ for all $\psi\langle a\rangle \in \mathcal{D}(\varphi\langle a\rangle)$, we get $\in_\delta \models \varphi'(c)\langle b/a\rangle$, for all $c \in U$, by the induction hypothesis, and thus $\in_\delta \models \varphi\langle b/a\rangle$.

- If φ is $\exists x\varphi'(x)$, then we have $\alpha_{\varphi'(c)} = \alpha_\varphi$, and thus $\mathcal{D}(\varphi'(c)\langle a\rangle) \subseteq \mathcal{D}(\varphi\langle a\rangle)$, for at least one $c \in U$. We then proceed likewise.

⊣

The next lemma allows to specify determining sets for atomic formulas.

Lemma 6.2.2. *Any atomic formula $\varphi\langle a\rangle$ has a determining set $\mathcal{D}$ of degree $\leq \alpha_\varphi$ in which any atomic formula different from $\top$ is of the form $(c \in a)\langle a\rangle$, where the '$a$' on the right side of '$\in$' is colored. Furthermore, if $\varphi\langle a\rangle$ is not itself of this form, then $\mathcal{D}$ is of degree $\leq \alpha_\varphi - 1$.*

Proof. The proof goes by transfinite induction on α_φ for atomic formulas $\varphi\langle a\rangle$. Obviously, if $\varphi\langle a\rangle$ is itself of the desired form or is reduced to $\top$, then $\mathcal{D}(\varphi\langle a\rangle) = \{\varphi\langle a\rangle\}$ is a determining set of degree $\leq \alpha_\varphi$, as requested. We then assume $\varphi\langle a\rangle$ to be of form $c \in \{x \mid \psi\}$ where some occurrences (maybe none) of a in c or $\psi(x)$ are colored. Notice that $\alpha_\varphi = \alpha_{\psi(c)} + 1 \geqslant 1$. Let $\psi(c)\langle a\rangle$ denote the formula $\psi(c)$ together with the coloring induced by those of c and ψ. We now look at $\mathcal{D}(\psi(c)\langle a\rangle)$, which is of degree $\leq \alpha_\varphi - 1$, and let $\mathcal{S}$ be the subset of it gathering the atomic formulas that are not of the desired form. By the induction hypothesis, for each $\chi\langle a\rangle \in \mathcal{S}$, let $\mathcal{D}_{\chi\langle a\rangle}$ be a suitable determining set for $\chi\langle a\rangle$. It is then easy to see that

$$\mathcal{D} := \left(\mathcal{D}(\psi(c)\langle a\rangle)\backslash\mathcal{S}\right) \cup \bigcup_{\chi\langle a\rangle\in\mathcal{S}} \mathcal{D}_{\chi\langle a\rangle}$$

is a suitable determining set of degree $\leq \alpha_\varphi - 1$ for $\psi(c)\langle a\rangle$, and so for $\varphi\langle a\rangle$ as well. ⊣

The following lemma ensures that those specific atomic formulas have the substitutivity property with respect to $\mathrel{\underset{\cdot}{=}}$.

Lemma 6.2.3. *Let $\varphi\langle a\rangle$ be an atomic formula of the form $(c \in a)\langle a\rangle$ where the 'a' on the right side of '$\in$' is colored. Recall that we suppose $\in_\delta\models \varphi\langle a\rangle$. Then $\in_\delta\models a \mathrel{\underset{\cdot}{=}} b \;\Rightarrow\; \in_\delta\models \varphi\langle b/a\rangle$*

Proof. Again the proof is by transfinite induction on α_φ. Assume $\in_\delta\models a \mathrel{\underset{\cdot}{=}} b$. Let $\varphi\langle a\rangle$ be an atomic formula of the form $(c \in a)\langle a\rangle$, where the '$a$' on the right side of '$\in$' is colored, and let $c\langle a\rangle$ denote c with the coloring induced by the one of $\varphi\langle a\rangle$. We then define $\varphi\langle a\rangle'$ as being $c\langle a\rangle \in a$, namely it is the same as $\varphi\langle a\rangle$ except that the 'a' on the right side of '$\in$' is not colored. Now let $\mathcal{D}$ be a determining set of degree $\leq \alpha_\varphi - 1$ for $\varphi\langle a\rangle'$ as provided by Lemma 6.2.2. We thus may apply the induction hypothesis to each atomic formula of the form $(c' \in a)\langle a\rangle$ that occurs in $\mathcal{D}$ to get $\in_\delta\models (c' \in a)\langle b/a\rangle$. Observe that if $\alpha_\varphi = 1$, then $\mathcal{D}$ is reduced to $\{\top\}$ and there is nothing to do. So, in any case, we have $\in_\delta\models \psi\langle b/a\rangle$ for each $\psi\langle a\rangle \in \mathcal{D}$. As $\mathcal{D}$ is a determining set for $\varphi\langle a\rangle'$, that yields $\in_\delta\models \varphi\langle b/a\rangle'$, namely $\in_\delta\models (c\langle b/a\rangle \in a)$. Since $\in_\delta\models a \mathrel{\underset{\cdot}{=}} b$, it follows that $\in_\delta\models (c\langle b/a\rangle \in b)$, and this latter is nothing but $\in_\delta\models \varphi\langle b/a\rangle$. ⊣

We are now ready to prove the result we announced.

Theorem 6.2.4. *For any $a, b \in U$, $\in_\delta\models a \mathrel{\underset{\cdot}{=}} b \;\Rightarrow\; \in_\delta\models a \doteq b$.*

Proof. Let $a, b \in U$ such that $\in_\delta\models a \mathrel{\underset{\cdot}{=}} b$ and let $d \in U$ with $\in_\delta\models a \in d$. We have to show that $\in_\delta\models b \in d$. Then let us denote by $\varphi\langle a\rangle$ the atomic formula $a \in d$ in which the only colored occurrence of 'a' is that one on the left side of '$\in$'. According to Lemma 6.2.2, $\varphi\langle a\rangle$ has a determining set $\mathcal{D}$ in which each atomic formula different from $\top$ is of the form $(c \in a)\langle a\rangle$, where the '$a$' on the right side of '$\in$' is colored. Now, by Lemma 6.2.3 and the assumption $\in_\delta\models a \mathrel{\underset{\cdot}{=}} b$, we get $\in_\delta\models (c \in a)\langle b/a\rangle$ for these formulas. As $\mathcal{D}$ is a determining set for $\varphi\langle a\rangle$, it follows that $\in_\delta\models \varphi\langle b/a\rangle$, and this is $\in_\delta\models b \in d$. ⊣

6.3 The quotient structure(s)

We have thus shown that $\emptyset_* \models {}^\flat Ext$, and it follows that $\dot{=}$ has the substitutivity property in $\langle U; \emptyset_*\rangle$ for $\mathscr{L}_*$-formulas. But because $\emptyset_* \models Abst[\mathscr{L}^+_{\tau*}]$, this can actually be extended to $\mathscr{L}^+_{\tau*}$-formulas, as the next observation shows.

Fact 6.3.1. Let $\tau(\bar{p})$ be any $\mathscr{L}^+_{\tau*}$-term, and let $\bar{a}$ and $\bar{b}$ in U, both of the same length as $\bar{p}$, such that $\emptyset_* \models a_k \dot{=} b_k$ for all k. Then $\emptyset_* \models \tau(\bar{a}) \dot{=} \tau(\bar{b})$.

Proof. We proceed by induction on the complexity of $\tau(\bar{p})$. Let $\tau(\bar{p})$ be $\{x \mid \varphi\}(\bar{p})$ for $\varphi(x, \bar{p})$ in $\mathscr{L}^+_{\tau*}$. From Theorem 6.2.4 and the induction hypothesis, it is easy to see that, for all $c \in U$, $\emptyset_* \models \varphi(c, \bar{a}) \leftrightarrow \varphi(c, \bar{b})$. Now, since $\emptyset_* \models Abst[\mathscr{L}^+_{\tau*}]$, it follows that $\in_\delta \models \tau(\bar{a}) \dot{=} \tau(\bar{b})$. ⊣

It follows therefrom that $\dot{=}$ is an acceptable interpretation of $=$ in $\langle U; \emptyset_*\rangle$. Consequently, this latter can be contracted to a *normal* countable model of $Abst[\mathscr{L}^+_{\tau*}] + Ext$ in which $W \notin W$. Likewise, $\langle U; \emptyset^*\rangle$ gives rise to a normal (dual) model of $Abst[\mathscr{L}^+_{\tau*}] + Ext$ in which $W \in W$.

It is natural to enquire whether these are isomorphic or not to the spine models we studied in Chapter 5. In trying to answer this question we surprisingly discovered that $Abst[\mathscr{L}^+_{\tau*}] + Ext$ has quantifier elimination.

Theorem 6.3.1. *For every $\mathscr{L}^+_{\tau*}$-formula φ, there exists a quantifier-free $\mathscr{L}^+_{\tau*}$-formula φ' such that $Abst[\mathscr{L}^+_{\tau*}] + Ext \vdash \varphi \leftrightarrow \varphi'$.*

Proof. This is in fact a consequence of Proposition 4.5.6, according to which, we recall, $Abst[\mathscr{L}^+_{\tau*}] + Ext \vdash \forall x \forall y (x \,\dot{\leqslant}\, y \to x \,\dot{\lessgtr}\, y)$. It follows therefrom that for any $\mathscr{L}^+_{\tau*}$-formula $\psi(x)$, we have $\forall x \forall y (x \,\dot{\leqslant}\, y \to (\psi(x) \to \psi(y)))$ (consider the term $\{x \mid \psi\}$), from which it is easily seen that $\exists x \psi(x) \leftrightarrow \psi(\mathrm{V})$ and $\forall x \psi(x) \leftrightarrow \psi(\Lambda)$. We can accordingly eliminate the quantifiers of any formula written in prenex form. ⊣

Remark 6.3.1. In light of the proof of Theorem 6.3.1, we can now see that the induction steps 4) & 5) in the proof of Theorem 5.3.4 were just manifestations of the equivalences $\exists x \psi(x) \leftrightarrow \psi(\mathrm{V})$ and $\forall x \psi(x) \leftrightarrow \psi(\Lambda)$.

Corollary 6.3.2. *For every $\mathscr{L}^+_{\tau*}$-formula φ, there exists a quantifier-free $\mathscr{L}^+_{\tau*}$-formula φ' such that $Abst[\mathscr{L}^+_{\tau*}] + Ext \vdash \{x \mid \varphi\} \dot{=} \{x \mid \varphi'\}$.*

Now, if one feels that the quotient structure(s) we have described here are really isomorphic to the Spine Model(s) of Chapter 5, then one has to show that for each *quantifier-free* $\mathscr{L}^+_{\tau*}$-formula $\varphi(x)$, either $\{x \mid \varphi\} \dot{=} \mathrm{W}$ or there is some $n \in \mathbb{N}$ such that $\{x \mid \varphi\} \dot{=} \mathscr{B}^n(\mathrm{V})$ or $\{x \mid \varphi\} \dot{=} \mathscr{B}^n(\Lambda)$.

This seems very unlikely to be derivable from $Abst[\mathscr{L}_{\tau*}^{+}] + Ext$; and a proof of this for $\langle U; \emptyset_* \rangle$ would probably require a subtle induction procedure as in 6.2, though we have not made any serious attempt in that direction yet.

At least, any answer, whether positive or negative, will certainly tell us a bit more about the real expressive power of $Abst[\mathscr{L}_{\tau*}^{+}] + Ext$, in comparison with the one of $Comp[\mathscr{L}^{[+]}] + Ext$ for instance, as initiated in Section 5.4. Just to be convinced, once more, that *the equality makes all the difference.*

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